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FOUNDATIONS

★**Suranyi, Janos. Reduction of the decision problem to formulas containing a bounded number of quantifiers only.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 759–762 (1949).

In three theorems the author presents different new reduction types, that is, classes of formulas to one of which every formula of the first order predicate calculus is equivalent. His first type is the class of binary formulas of the form $(x_1)(x_2)(x_3)N_1$ and $(x_1)(x_2)(Ex_4)N_2$. Equivalent prenex forms are $(x_1)(x_2)(x_3)(Ex_4)N$ and $(x_1)(x_2)(Ex_4)(x_3)N$. The next type consists of binary formulas of the form $(x)(Ey)(Ez)(u)(v)P$ and $(Ey)(x)(Ez)(u)(v)P'$, with not more than seven binary relations in the matrix. The third type consists of binary formulas of the form $(x)(Ey)(Ez_1)(Ez_2)(Ez_3)(Ez_4)(u)Q$ and $(Ey)(x)(Ez_1)(Ez_2)(Ez_3)(Ez_4)(u)Q'$, with not more than four binary relations, including the identity relation, in the matrix. These formulas have each a fixed number of quantifiers. Reduction types found by other authors allow an unspecified number of quantifiers. The paper includes sketches of the proofs of these theorems. *O. Frink.*

Kalmár, László, and Surányi, János. On the reduction of the decision problem. II. Gödel prefix, a single binary predicate. J. Symbolic Logic 12, 65–73 (1947).

[For part I, see Kalmár, same J. 4, 1–9 (1939).] The authors improve on a reduction theorem of Gödel by showing that, given any formula of the first order predicate calculus, it is possible to construct an equivalent formula with prefix of the form $(x_1)(x_2)(x_3)(Ex_4)\cdots(Ex_n)$, and containing no predicate variable except a single binary one. The last condition constitutes the improvement, since Gödel's original theorem allowed more than one binary predicate variable. *O. Frink* (State College, Pa.).

Rasiowa, H. Axiomatisation d'un système partiel de la théorie de la déduction. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 40 (1947), 22–37 (1948). (French. Polish summary)

Let Epq and E$'pq$ mean, respectively, that p and q are equivalent and that p and q are non-equivalent. The author considers those propositions which can be expressed in terms of E and E$'$ alone, and shows that all theorems of this type can be derived from the following two axioms: (A.I.) EEpqEErqEpr, (A.II.) EEpqEE$'rq$E$'pr$. If only theorems expressed in terms of E alone are considered, then (A.I.) is sufficient. *B. Jónsson* (Providence, R. I.).

Rasiowa, H. Sur un certain système d'axiomes du calcul des propositions. Norsk Mat. Tidsskr. 31, 1–3 (1949).

E. Götlind has shown [same Tidsskr. 29, 1–4 (1947); these Rev. 9, 1] that the following axioms are sufficient for the calculus of propositions: (a) $p\vee p\to p$, (b) $p\to p\vee q$, (c) $p\to p$, (d) $p\to r.\to:q\vee p.\to.r\vee q$. He asked whether (c) could be derived from the remaining axioms. The author gives an affirmative answer to this question, thus showing that (a), (b), and (d) constitute a complete as well as independent set of axioms. *B. Jónsson* (Providence, R. I.).

Rose, Alan. A reduction in the number of the axioms of the propositional calculus. Norsk Mat. Tidsskr. 31, 113–115 (1949).

It is shown that the following axioms are sufficient for the propositional calculus: (a) $X\vee\overline{(X\vee X)}$, (b) $X\to(X\vee Y)$, (c) $(Y\vee\bar{X})\to(Z\vee X\to Y\vee Z)$. *B. Jónsson.*

Skolem, Th. Remark on the articles of H. Rasiowa and A. Rose in this volume. Norsk Mat. Tidsskr. 31, 115 (1949). (Norwegian)

Cf. the two preceding reviews.

Bergmann, Gustav. A syntactical characterization of S5. J. Symbolic Logic 14, 173–174 (1949).

A condition is given in order that a syntactically constructed system of modal logic lead to the Lewis system S5. [The hypothesis P used by the author can be shown to be superfluous: cf. the review in the same vol., 260 (1949).] *J. C. C. McKinsey* (Santa Monica, Calif.).

Bergmann, Gustav. The finite representations of S5. Methodos 1, 217–219 (1949).

It is shown that every finite normal matrix for the Lewis system S5 is either a matrix of the type described by Henle (i.e., satisfies the condition $\Diamond x=1$ for all $x\neq 0$), or is isomorphic to a direct union of such matrices. *J. C. C. McKinsey* (Santa Monica, Calif.).

Halldén, Sören. Results concerning the decision problem of Lewis's calculi S3 and S6. J. Symbolic Logic 14, 230–236 (1950).

By S6 and S7 are meant the systems of modal sentential calculus obtained by adding $\Diamond\Diamond p$ to the Lewis systems S2 and S3, respectively; by S8 is meant the system obtained by adding $\sim\Diamond\sim\Diamond\Diamond p$ to S3. The author gives a decision method for S6. He shows that a formula is provable in S3 if and only if it is provable both in S4 and S7; in view of the fact that a decision method is known for S4, this reduces the decision problem for S3 to the decision problem for S7. He also shows that the number of complete extensions of S3 is greater by one than the number of complete extensions of S7; and that a formula is provable in S3 if and only if it is provable both in S8 and in the system obtained by adding to S3 the axiom $\Diamond[\sim\Diamond\sim p \prec \sim\Diamond\sim\sim\Diamond\sim p]$. Some of the lemmas used in the proofs appear interesting in themselves: these lemmas formulate properties of fragments of the classical sentential calculus, and their extensions. *J. C. C. McKinsey* (Santa Monica, Calif.).

Halldén, Sören. A question concerning a logical calculus related to Lewis' system of strict implication, which is of special interest for the study of entailment. Theoria **14**, 265–269 (1948).

The author proposes a system S_0 of propositional algebra which differs from the system S_1 of Lewis [C. I. Lewis and C. H. Langford, Symbolic Logic, Century, New York-London, 1932, p. 500] in that strict implication is taken as primitive rather than defined in terms of necessity. This is weaker than S_1 in that it admits the interpretation where $\sim p$ has the same value as p. A list is given of the theorems which were proved by Lewis for S_1 and are not true for S_0; this list the author believes to be complete.

H. B. Curry (State College, Pa.).

Aubert, Karl Egil. Relations généralisées et indépendance logique des notions de réflexivité, symétrie et transitivité. C. R. Acad. Sci. Paris **229**, 284–286, 538–540 (1949).

The author considers extensions of the binary relations in the following general way. A set E is given. An n-uple $(x_1, \cdots, x_n)$, $x_i \epsilon E$, is any element in the product space E^n; an uple is any n-uple for some n. A relation is defined as a correspondence of all uples to some truth set $\mathfrak{C}$ whose cardinal number may be arbitrary, not less than 2. There are concepts of reflexivity $(x, x, \cdots, x)$ and symmetry $(x_1, \cdots, x_n) = (\sigma x_1, \cdots, \sigma x_n)$, where σ is some permutation in the symmetric group on n elements. A wide generalisation of transitivity is also introduced. The author studies particularly the case where $\mathfrak{C}$ corresponds to the truth-values in a many-valued logic, for instance, the system of Łukasiewicz.

In the second note the author establishes necessary and sufficient conditions for the logical independence of these three concepts and their negations under certain general conditions. Instead of the matrix representation in the binary case the uples are ordered in sequences corresponding to a second sequence of truth values defined by and defining the relation.

O. Ore (New Haven, Conn.).

Saarnio, Uuno. Über die Konverse der Relation. Filosofisen Yhdistykse Vuosikirja **15**, 167–176 (1948).

The author distinguishes between two meanings of the term "converse of a relation," and illustrates the difference with examples and diagrams for certain symmetric, unsymmetric, asymmetric, reflexive and irreflexive relations.

O. Frink (State College, Pa.).

Valen-Sendstad, Olav. Der Wahrheitsbegriff in der zweiwertigen Logik. Theoria **15**, 367–383 (1949).

This is a metaphysical, and, at least to the reviewer, obscure critique of the basic notions of propositional algebra. There is no mathematical content.

H. B. Curry.

Novikov, P. S. On the axiom of complete induction. Doklady Akad. Nauk SSSR (N.S.) **64**, 457–459 (1949). (Russian)

The paper is concerned with a characterization of a class of arithmetical formulas. These are the formulas derived by logical calculus (presumably classical first order predicate calculus) from the axioms for order and equality, together with the schemes of primitive recursive definition, without the use of complete induction. The argument depends on a previous paper [same vol., 293–295 (1949); these Rev. **11**, 1] and is not intelligible without it.

H. B. Curry (State College, Pa.).

Novikov, P. S. The consistency of certain statements of the theory of sets. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 171 (1949). (Russian)

Consistency proofs are announced for the adjunction of certain hypotheses on projective sets to the Gödel-Bernays set-theory ("G.B."). Let "sets P_n" and "sets CP_n" denote projective sets of order n and their complements, respectively. [The underlying space is not specified.] (A) Proofs will be supplied for two theorems announced by Gödel [Proc. Nat. Acad. Sci. U. S. A. **24**, 556–557 (1938)], namely, consistency in G.B. of the existence of noncountable totally imperfect sets CP_1, and of nonmeasurable sets P_2. (B) Consistency in G.B. of the following will be proved. Let $F(P_n)$ be the maximal field of sets in P_n. For all sufficiently large n, two disjoint sets CP_n are separated by sets of $F(P_n)$, but disjoint sets P_n are not in general so separated. The method of proof will be to establish the assertions in Gödel's "Δ-model" [Gödel, The Consistency of the Continuum Hypothesis, Princeton University Press, 1940; these Rev. **2**, 66].

M. H. A. Newman (Manchester).

Zich, O. V. A contribution to the theory of integers and one-to-one transformations. Rozpravy II. Třídy České Akad. **58**, no. 11, 12 pp. (1948). (Czech)

The author develops a theory of symbolic logic which, he states, cannot be understood as a play with symbols. He emphasizes that in his theory the signs always have a meaning; they correspond to the objects of the outside world and to their real relations or to the mental constructions which are built up according to their images. As there are never two objects which are identical in every respect, the author uses the same sign for the same object under the same circumstances, but different signs for different objects. In the notation of the author $(\underline{Ex})(Ey)f(x, y)$ has the same meaning as $(Ex)(Ey)[f(x, y) \,\&\, \overline{x=y}]$ in Hilbert and Ackermann's Grundzüge der theoretischen Logik. The author agrees with Wittgenstein in rejecting the identity sign as an essential constituent of logical notation. He develops a logical theory without using the Leibnitz definition of identity [in the notation of Hilbert and Ackermann $x=y=_{\mathrm{Df}}(f)(f(x)\rightarrow f(y))$] and also without using identity as an undefined fundamental idea as Carnap proposes. The author's objection to Leibnitz's conception of the identity relation is the latter's use of the operator (f), as it would be impossible to verify the fact that qualities of x must also be qualities of y. To derive the theory of integers and of one-to-one transformations of sets which have no common elements, the author uses the following definition of equivalence:

$$\mathrm{Ekv}\ (f, g) =_{\mathrm{Df}} [f(x) \rightleftarrows g(x)]$$

but no general definition of identity. His system of axioms is similar to Hilbert and Ackermann's, but it contains additional rules for the use of operators, for example (VI A): In brackets following the operator signs (x) or $(\underline{Ex})$, a sign different from x means an object different from x. The formulation of certain other axioms is therefore different from Hilbert and Ackermann's.

The author defines the integers without using the identity sign. For example, he defines the number 1 as follows:

$$1(F) =_{\mathrm{Df}} (Ex)[F(x) \& \overline{(\underline{Ey})}(F(x) \& F(y))],$$

where (according to (VI A)) the x must be different from y. The fact that there is a one-to-one transformation between

two sets I and K belonging to the same number (relation $M(I, K)$) is defined without the use of the symbol of elementary identity:

$$M(I, K) =_{Df} (ER)\{(x)[I(x) \to (Ey)(R(x, y) \,\&\, K(y) \,\&\, (\overline{Et})(R(x, t) \,\&\, K(t)))] \,\&\, (y)[K(y) \to (Ex)(R(x, y) \,\&\, I(x) \,\&\, (\overline{Eu})(R(u, y) \,\&\, I(u)))]\}.$$

In order to prove the commutativity and the transitivity of M the author assumes that I, K, L have no elements in common. *J. Mayer* (New York, N. Y.).

Denjoy, Arnaud. Récurrence et antirécurrence. C. R. Acad. Sci. Paris **229**, 637–640 (1949).

A brief discussion of the recursive definition of natural numbers, and a definition of "antirecursive definitions."
I. L. Novak (Wellesley, Mass.).

Brouwer, L.-E.-J. Remarques sur la notion d'ordre. C. R. Acad. Sci. Paris **230**, 263–265 (1950).

A definition for partial ordering is presented and intuitionistic distinctions among ordered species are outlined. A partial ordering is said to be discrete in case, for any two elements a and b, either $a \sim b$ (a is of the same level as b) or not $a \sim b$. It is said to be quasicomplete in case the absurdity of $a \sim b$ implies either $a<b$ or $a>b$. When both discrete and quasicomplete, an ordering is said to be complete. A partial ordering is said to be virtual if it satisfies the conditions: (i) if not $a \sim b$ and not $a<b$, then $a>b$; (ii) if not $a>b$ and not $a<b$, then $a \sim b$. The natural ordering of the intuitionistic continuum is an example of virtual order. Complete order implies virtual order. The example of an essentially negative property [Nederl. Akad. Wetensch., Proc. **51**, 963–964 = Indagationes Math. **10**, 322–323 (1948); these Rev. **10**, 421] is cited to show that virtual order does not imply quasicomplete order. A further example, also depending on the intuitionistic continuum, shows that quasicomplete order does not imply virtual order. A simpler example of the independence of these two concepts is also presented. *D. Nelson* (Washington, D. C.).

Brouwer, L.-E.-J. Sur la possibilité d'ordonner le continu. C. R. Acad. Sci. Paris **230**, 349–350 (1950).

The counter-example cited in the paper reviewed above showed that the natural ordering of the intuitionistic continuum is not quasicomplete, i.e., it is not the case that for any two nonequal real numbers one must be greater than the other. The present note shows that there exists no quasicomplete ordering of the intuitionistic continuum in which the relation of equality of level is interpreted as equality of real numbers. *D. Nelson.*

Loonstra, F. The concept of "order" in mathematics. Math. Centrum Amsterdam. Rapport **ZW-1949-014**, 4 pp. (1949). (Dutch)

Dequoy, Nicole. Exposé d'un type de raisonnement en mathématique intuitioniste sans négation et résultats obtenus pour la géométrie projective plane. C. R. Acad. Sci. Paris **230**, 357–359 (1950).

The author continues her development of projective geometry in a negationless intuitionistic logic [same C. R. **228**, 1098–1100 (1949); these Rev. **10**, 499]. It is observed that applications of an intuitionistically valid form of reductio ad absurdum used by Heyting in his intuitionistic projective geometry [Math. Ann. **98**, 491–538 (1927)] may be replaced with a set-theoretic argument depending on negationless axioms for distinctness. Specific elementary theorems which the author has established by this method are stated.
D. Nelson (Washington, D. C.).

Bockstaele, Paul. Intuitionism among the French mathematicians. Verh. Vlaamsche Acad. Kl. Wetensch. **11**, no. 32, 123 pp. (1949). (Dutch)

de Bengy-Puyvallée, Renaud. Sur les relations d'incomposabilité dans les logiques de complémentarité. C. R. Acad. Sci. Paris **230**, 265–267 (1950).

This note consists of a series of definitions of various logical notions which are intended to be applied in quantum theory. The definitions are obscure, however, because of apparent carelessness in formulation: thus, for example, in one case a free variable occurs in the definiens which does not occur in the definiendum; in another case, a letter occurs as a free variable in the definiendum but only as a bound variable in the definiens. No nontrivial conclusions are drawn from the definitions given, and no serious attempt is made to relate them in a precise way to physical science.
J. C. C. McKinsey (Santa Monica, Calif.).

***Reboul, Georges, et Reboul, Jean-Antoine. Un Axiome Universel. Ses Applications aux Sciences Expérimentales.** Gauthier-Villars, Paris, 1950. xx+148 pp. 1300 francs.

Any differentiable law of nature, when written in the form of a total differential $\sum \beta_i(dU_i)/U_i$, may be interpreted as a combination of weighted probabilities. However, the interpretation is purely formal and apparently leads to no new results. *C. C. Torrance* (Annapolis, Md.).

Lefschetz, S. The structure of mathematics. American Scientist **38**, 105–111 (1950).

***Wiener, Norbert. Entropy and information.** Proc. Symposia Appl. Math. **2**, 89 (1950). $3.00.

ALGEBRA

Peremans, W., and Kemperman, J. Numbering problem of S. Dockx. Math. Centrum Amsterdam. Rapport **ZW-1949-005**, 4 pp. (1949). (Dutch)

The problem solved is that of the relationship of two kinds of numbers assigned to the cells of a square V_k having $(2k)^2$ cells; e.g., for $k=2$,

$$\begin{array}{cccc} 6,0 & 7,-1 & 8,-1 & 9,0 \\ 5,1 & 11,0 & 12,0 & 10,1 \\ 18,-1 & 20,0 & 19,0 & 13,-1 \\ 17,0 & 16,1 & 15,1 & 14,0. \end{array}$$

The numbers of the first kind run from $a_{k-1}+1$ to a_k where $a_k = 2k(k+1)(2k+1)/3$ and, as will be noticed, are put on serially in strips, running from left to right in the upper half and from right to left in the lower, each strip containing the outermost cells not contained in any preceding strip. The numbers of the second kind are essentially given by the upper left quarter whose mode of generation is

$$0 \qquad \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \qquad \begin{array}{ccc} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{array} \qquad \cdots.$$

With respect to these numbers the square has mirror symmetry about its central vertical axis and symmetry with a factor -1 about its central horizontal axis. A formula, too detailed to quote, gives numbers of the second kind as functions of those of the first kind. *J. Riordan.*

Münzner, Hans. Über die Verteilungszahl. Arch. Math. **2**, 42–48 (1949).

The numbers concerned are those enumerating distributions of N objects, a_i of which are of the ith kind, into N cells, b_j of which are of the jth kind; they are denoted by $D_N(a_1, a_2, \cdots, a_n/b_1, b_2, \cdots, b_m)$. By a probability argument, using the central limit theorem, the following asymptotic expression is derived:

$$D_N \sim m^{m/2}(A/(2\pi N))^{(m-1)/2}\prod_1^n \binom{a_i+m-1}{m-1} e^{-A\chi^2/2}$$

with $A = N(m+1)(Nm+\sum a_i^2)^{-1}$, $\chi^2 = (m/N)\sum(b_j - N/m)^2$. It is noted that for close accuracy it is desirable that χ^2 be small (less than m). *J. Riordan* (New York, N. Y.).

✱Sade, Albert. Sur les suites hautes des permutations. Published by the author, Marseille, 1949. 12 pp.

A "suite haute" of a permutation is a reading left to right of elements in natural order; e.g., 31425 has the two readings 12, 345. In the first part of this pamphlet, the author studies the enumeration of permutations by number of readings and finds the number with x readings, $F(n, x)$, to be given by $\sum_0^x(-1)^j\binom{n+1}{j}(x-j)^n$, a result known for enumeration by rising sequences [cf. von Schrutka, Math. Ann. **118**, 246–250 (1941); these Rev. **6**, 32] which, as the author notes, are related to readings through conjugacy of the corresponding permutations. But he fails to notice, or derive, the simple recurrence formula $F(n+1, x) = xF(n, x) + (n-x+2)F(n, x-1)$. In the second part the enumeration is by length of longest reading and the results are piecemeal: $U(n, r)$ being the number with longest reading r, $U(n, 1) = U(n, n) = 1$; $U(n, n-1) = 2(n-1)$, $n>3$; $U(n, n-2) = 3n^2-7n+1$, $n>4$; $U(n, n-3) = 4n^3-18n^2+16n$, $n>6$. A table is given for $n=1$ to 8. *J. Riordan* (New York, N. Y.).

✱Sade, Albert. Décomposition des locomotions en facteurs de classe haute donnée. Published by the author, Marseille, 1949. 8 pp.

The term "locomotion" is used for the inverse of a permutation considered as an operation, and the classes considered (classe haute) are those determined by the number of readings, left to right, needed to put a permutation in natural order; e.g., 35124 is of class 3 with readings 12, 34, 5. The terminology is apparently suggested by the train classification problem treated by R. Sprague [Math. Ann. **121**, 52–53 (1949); these Rev. **10**, 670]. It is shown by a constructive method that a locomotion of class $s=tu$ has a unique expression as a product of locomotions of classes t and u, respectively. The number of decompositions into locomotions of classes t and u, where $tu>s$, is also determined (by inclusion and exclusion). Finally the locomotions which permit decompositions into permutable factors, essentially arithmetic progressions with a proper modulus, are examined in detail. *J. Riordan* (New York, N. Y.).

Grant, Harold S. On a formula for circular permutations. Math. Mag. **23**, 133–136 (1950).

The formula derived, for enumerating the number of nonsensed circular permutations, is equivalent to that of Fu [Wu-Han Univ. J. Sci. **8**, no. 1, 1.1–1.16 (1942); these Rev. **8**, 365]. *J. Riordan* (New York, N. Y.).

Bose, R. C. A note on Fisher's inequality for balanced incomplete block designs. Ann. Math. Statistics **20**, 619–620 (1949).

A new and simple proof of the inequality for incomplete balanced block designs. *H. B. Mann* (Berkeley, Calif.).

Chowla, S., and Ryser, H. J. Combinatorial problems. Canadian J. Math. **2**, 93–99 (1950).

This is an extension of the results of Bruck and Ryser [same J. **1**, 88–93 (1949); these Rev. **10**, 319] on projective planes to symmetric incomplete block designs. In addition this theory is freed from its dependence on the Minkowski-Hasse theory of invariants of rational quadratic forms under rational transformations. In a design we require v elements in v sets such that every set contains k distinct elements and every pair of sets has exactly $\lambda = k(k-1)/(v-1)$ elements in common. It is shown here that if v is even and $k-\lambda$ is not a square then no design exists. For v odd it is shown that if $v\equiv 3 \pmod 4$ and if there is an odd prime p dividing $k-\lambda$ to an odd power such that $(-\lambda|p) = -1$ then no design exists. These are the first nontrivial results on existence of designs except for special cases. *M. Hall.*

Shrikhande, S. S. The impossibility of certain symmetrical balanced incomplete block designs. Ann. Math. Statistics **21**, 106–111 (1950).

Utilizing the methods of Bruck and Ryser [Canadian J. Math. **1**, 88–93 (1949); these Rev. **10**, 319] the author proves that a symmetrical incomplete balanced block design with parameters v, r, λ is possible only if

$$(r-\lambda, -1)_p{}^{v(v-1)/2}(r-\lambda, v)_p{}^{2v-3} = +1,$$

where $(m, n)_p$ is the Hilbert norm residue symbol and if v is even only if $r-\lambda$ is a perfect square. Both results were published almost simultaneously by Chowla and Ryser [see the preceding review], who obtained them by elementary methods. The second result was also obtained by Schützenberger [Ann. Eugenics **14**, 286–287 (1949); these Rev. **11**, 3]. The author evidently obtained both results independently. *H. B. Mann* (Berkeley, Calif.).

Lotan, Moshe. A problem in difference sets. Amer. Math. Monthly **56**, 535–541 (1949).

If a, b, c, d is a set of four real numbers, the set of four absolute differences $|a-b|$, $|b-c|$, $|c-d|$, $|d-a|$ is called the derivative of the original set. The author proves that if we form the successive derivatives of the set a, b, c, d we always obtain, after a finite number of steps, a set whose terms are all zero, except in the case where the first derivative of the original set is, apart from trivial transformations, of the form $1, q, q^2, q^3$, where q is the positive solution of the equation $q^3-q^2-q-1=0$. *W. H. Gage.*

Richter, H. Ein einfacher Beweis der Newtonschen und der Waringschen Formel für die Potenzsummen. Arch. Math. **2**, 1–4 (1949).

Waring's formulae, expressing the sums of powers of n quantities in terms of their elementary symmetric functions and vice versa, are proved by a simple use of generating functions. *W. Ledermann* (Manchester).

Kesava Menon, P. On the homogeneous cubic equation. Math. Student **16** (1948), 20–24 (1949).

Narasinga Rao, A. On the homogeneous cubic equation—Geometrical approach. Math. Student 16 (1948), 25–27 (1949).

Taussky, Olga. A recurring theorem on determinants. Amer. Math. Monthly 56, 672–676 (1949).

Let A_i denote the sum of the moduli of the nondiagonal terms of the ith row of the matrix (a_{ik}). The basic theorem in this paper on determinants with dominant principal diagonal is the following. If (a_{ik}) is an $n\times n$ matrix with complex elements such that $|a_{ii}|>A_i$ for $i=1, 2, \cdots, n$, then the determinant $|a_{ik}|$ is not zero. After giving a simple proof of this theorem, the author treats its extensions, specializations to real matrices, generalizations, and applications to the location of the characteristic roots of matrices. The following theorems are further samples of the results. (1) If (a_{ik}) is a real matrix such that $a_{ii}>A_i$ for $i=1, 2, \cdots, n$, then $|a_{ik}|>0$. (2) The characteristic roots of any $n\times n$ matrix (a_{ik}) with complex coefficients lie inside the circles with centers a_{ii} and radii A_i; a boundary point can only be a characteristic root if it is also on the boundary of the $n-1$ other circles. The results are not new, but they have been rediscovered and reported as new many times. There is an extensive bibliography which traces the development of the results from 1881. *G. B. Price* (Lawrence, Kan.).

Coulson, C. A. Notes on the secular determinant in molecular orbital theory. Proc. Cambridge Philos. Soc. 46, 202–205 (1950).

The author considers the expansion of the secular determinant

$$\Delta(x)\equiv\begin{vmatrix} x & \beta_{12} & \cdots & \beta_{1n}\\ \beta_{21} & x & \cdots & \beta_{2n}\\ \cdots & & \cdots & \\ \beta_{n1} & \beta_{n2} & \cdots & x\end{vmatrix}=0$$

for a certain class of hydrocarbon molecules and obtains an interpretation of this expansion in terms of the number of bonds in the molecule which permits its calculation from knowledge of the molecule. A theorem on determinants with dominant principal diagonal [see the preceding review] is applied to obtain bounds for the roots of $\Delta(x)=0$. Finally, if $\psi=c_1\varphi_1+c_2\varphi_2+\cdots+c_n\varphi_n$ is the lowest molecular orbital, $\varphi_1, \cdots, \varphi_n$ being the usual atomic orbitals, then it is shown that all the coefficients c_r have the same sign, and that this function is the only one without changes of sign in the coefficients. *G. B. Price* (Lawrence, Kan.).

Wielandt, Helmut. Die Einschliessung von Eigenwerten normaler Matrizen. Math. Ann. 121, 234–241 (1949).

Call the complex numbers $\lambda_1, \cdots, \lambda_n$ a spectrum consistent with the non-proportional column vectors y and z if there exists an nth order normal matrix L having the λ_i for its characteristic values and such that $Ly=z$. Let the λ_i be projected stereographically from the Gaussian plane onto a sphere K with center at $\rho=(m_{01}/m_{00})^{\frac{1}{2}}$ and radius $\delta=(m_{11}/m_{00}-\rho\bar{\rho})^{\frac{1}{2}}$, where $m_{00}=(y, y)$, $m_{01}=(z, y)=\sum\bar{\eta}_i\zeta_i$ and $m_{11}=(z, z)$. By an inclusion class is meant any subset of K which necessarily contains at least one λ_i from every spectrum consistent with y and z, and by an exact half sphere of K is meant a closed half sphere with a half closed semicircle deleted from its boundary. All possible inclusion classes are now determined for $n>3$: a subset of K is an inclusion class if and only if it contains an exact half sphere. For $n=3$ an example shows that other inclusion classes exist. If L is specialized further (unitary, Hermitian, real symmetric) correspondingly sharpened results are obtained. *W. Givens* (Knoxville, Tenn.).

Parodi, Maurice. Sur les limites des modules des racines des équations algébriques. Bull. Sci. Math. (2) 73, 135–144 (1949).

This is a summary of known bounds for characteristic roots of matrices. Bounds for the roots of arbitrary algebraic equations are deduced by interpreting them as characteristic equations of matrices. Numerical examples are studied; they indicate that the results of A. Brauer [Duke Math. J. 13, 387–395 (1946); 14, 21–26 (1947); these Rev. 8, 192, 559] lead to the sharpest bounds. *O. Todd-Taussky.*

Parodi, Maurice. Contribution à l'étude de la stabilité. J. Phys. Radium (8) 10, 348–352 (1949).

The methods already introduced by the author [see C. R. Acad. Sci. Paris 228, 1400–1402 (1949); Bull. Sci. Math. (2) 72, 136–138 (1948); these Rev. 10, 671] for the study of the roots of stability equations are discussed and illustrated by numerical examples. *O. Todd-Taussky.*

Vakselj, Anton. Le calcul des matrices des substitutions linéaires du 2ème ordre. Akad. Ljubljani. Mat.-Prirodoslov. Razred. Mat. Odsek. Razprave 3, 21–39 (1947). (Slovenian. French summary)

For two square matrices of the second order S_1, S_2 this paper defines a polar product, which is a matrix of the second order with elements bilinear in those of S_1, S_2 but distinct from the usual matrix product, and an axial product, which is a scalar. The transform of a polar product is the polar product of the transforms. Various products combining ordinary, polar and axial multiplication are formed. Some of the formulas present analogies with the vector calculus. *J. M. Thomas* (Durham, N. C.).

Kofink, W. Zur Mathematik der Diracmatrizen: Die Bargmannsche Hermitisierungsmatrix A und die Paulische Transpositionsmatrix B. Math. Z. 51, 702–711 (1949).

The author treats the problem of finding matrices A and B which respectively transform the Dirac matrices into their Hermitian conjugates and into their transposes. Certain special cases are also considered. *R. M. Thrall.*

⁎Woodbury, Max A. The Stability of Output-Input Matrices. Chicago, Ill., 1949. 5 pp.

Die Matrizen X, Y seien benachbart: $Y=X+D$. Dann bekommt man die Reziproke Y^{-1} aus X^{-1} mittels: $Y^{-1}=X^{-1}(E+DX^{-1})^{-1}$. Sind in D einige Zeilen gleich Null, so vereinfacht sich die Rechnung. *E. Bodewig.*

Smiley, M. F. The rational canonical form of a matrix. II. Amer. Math. Monthly 56, 542–544 (1949).

This is a continuation of a previous paper [same Monthly 49, 451–454 (1942); these Rev. 4, 68]. The two papers together give a simple and direct method of reducing a matrix to the rational canonical form. No use is made of invariant factors, and no special attention is necessary to take care of the derogatory case. *N. H. McCoy.*

Vinograde, B. Simultaneous reduction of quadratic forms. Proc. Iowa Acad. Sci. 55, 291–292 (1949).

The following theorem is proved. Let $A_1, \cdots, A_k$ be n-rowed real matrices of ranks $n_1, \cdots, n_k$, respectively, such that $A_1+\cdots+A_k$ is nonsingular and $n_1+\cdots+n_k=n$. Then

there exists a real nonsingular matrix Q which simultaneously transforms the k given matrices into diagonal form. More precisely

$$QA_iQ' = \mathrm{diag}\,(0, L_i, 0), \qquad i=1, \cdots, k,$$

where the first $n_1+\cdots+n_{i-1}$ and the last $n_{i+1}+\cdots+n_k$ diagonal elements are zero and where L_i is an n_i-rowed diagonal matrix whose elements are the nonzero characteristic roots of A_i. This result generalizes a theorem of G. C. Cochran [Proc. Cambridge Philos. Soc. **30**, 178–191 (1934)].

W. Ledermann (Manchester).

Lepage, Th. Sur les matrices symétriques et les modules de formes alternées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 325–345 (1949).

Let V be a $2n$-dimensional space over a field K of characteristic greater than n. Let A_n be the exterior algebra over V, and set $\Gamma=\sum_{i=1}^n x_iy_i$ (where $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ is a base of V). Every homogeneous element of A_n of degree $p\leqq n$ may be written in one and only one way in the form $r_p+r_{p-2}\Gamma+\cdots+r_{p-2q}\Gamma^q$, where each r_k is a form of degree k which satisfies the condition $r_k\Gamma^{n-k+1}=0$. It follows that the only symplectic invariants in A_n are the linear combinations of $1, \Gamma, \cdots, \Gamma^n$. The main object of the paper is the study of the spaces R_p, where R_p is composed of the forms r_p of degree p such that $r_p\Gamma^{n-p+1}=0$ (incidentally, these spaces yield irreducible representations of the symplectic group). It is proved that each R_p is spanned by "simple" forms (i.e., products of p elements of V). The case $p=n$ is considered in more detail. If P is any symmetric matrix (p_{ij}) of degree n, R_n contains $\sigma_P=\prod_{i=1}^n(y_i+p_{ij}x_j)$, and is spanned by the σ_P's which correspond to orthogonal symmetric matrices P with coefficients 0 or ±1. This result is applied to the study of the linear relations which bind the minors of all orders of an arbitrary symmetric matrix.

C. Chevalley (New York, N. Y.).

Lepage, Th. Sur les fonctions linéaires des mineurs d'une matrice symétrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 694–708 (1949).

Let $f(P)$ be a linear function with complex coefficients of the n-rowed minors of the $n\times 2n$ matrix (E, P), where E is the unit $n\times n$ matrix and $P=(p_{ij})$ is a symmetric $n\times n$ matrix. If $n=2$ then $f(P)=|A+BP|$ (determinant) where A, B are $n\times n$ matrices, and the differential equation $f(P)=0$ $(p_{ij}=\partial^2z/\partial x_i\partial x_j)$ has precisely one or two systems of characteristics according as the antisymmetric matrix $AB'-BA'$ has rank 0 or 2; for $n>2$ in general $f(P)$ cannot be written in this form and the differential equation does not have characteristics. The author studies the vector space of all such $f(P)$, which space is isomorphic with a subspace of a certain exterior algebra of degree $2n$. Among many other results he gives criteria in terms of certain exterior forms related to $f(P)$ that $f(P)$ be expressible in the form $|A+BP|$, and clarifies the problem of characteristics. Examples are given.

E. R. Kolchin (New York, N. Y.).

Papy, Georges. Sur les formes extérieures définies sur un anneau d'intégrité à idéaux principaux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 378–398 (1949).

Let I be a domain of integrity whose ideals are principal, and let L be a free module over I, with a base, of $2n$ elements $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$. Let E be the exterior algebra over L, and H the quadratic element $\sum_{i=1}^n x_iy_i$ of E; H defines a symplectic geometry on L. The author calls simple (rather than the usual "decomposable," which he uses in another meaning) an element F of E which is a product of elements of L. He proves that a simple element is symplectically equivalent to a monomial $ax_1y_1\cdots x_ty_ty_{2t+1}\cdots y_d$; the number $s=d-t$ is a symplectic invariant of the form, and so is the ideal Ia; moreover, if $d=2s$, a itself is a symplectic invariant. A homogeneous element of E is called separated if it is homogeneous in the x's and the y's separately; if F is simple and separated, then the reduction of F to normal form may be accomplished by a separated symplectic transformation, i.e., a transformation which leaves invariant the modules spanned by the x's and the y's separately. The author then considers the symplectic reduction of quadratic elements of E, a problem which is equivalent to that of the simultaneous reduction of two skew symmetric matrices. He proves the known theorem that the only invariants of a quadratic element F are the elementary divisors of $F-xH$, and gives reduced forms for F.

C. Chevalley (New York, N. Y.).

Makar, R. H. A note on power series in matrices. Quart. J. Math., Oxford Ser. **20**, 235–237 (1949).

The matrices considered are infinite matrices with complex elements. An associative field is a class of matrices closed under addition and multiplication by scalars, and such that all finite products of elements of the class exist and are associative. A self-associative matrix is a matrix which is an element of some associative field. An associative field is bounded if there is defined on elements B of the field a real-valued bound (or absolute value) $|B|$ having the usual properties. Now consider the series $\sum_{n=0}^\infty c_nA^n$, where A is assumed to be a self-associative matrix, and let ρ be the radius of convergence of the series $\sum_{n=0}^\infty c_nz^n$. The author shows that the matrix series is absolutely convergent provided there exist relatively prime positive integers r and s such that A^r and A^s are elements of a bounded associative field and $A^r<\rho^r$ (or $A^s<\rho^s$).

N. H. McCoy.

Abstract Algebra

★Hasse, Helmut. Algebra und Zahlentheorie. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 39–58. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

Shoda, Kenjiro. Allgemeine Algebra. Osaka Math. J. **1**, 182–225 (1949).

The author reviews and extends his previous investigations on general algebraic systems [Proc. Imp. Acad. Tokyo **17**, 323–327 (1941); **18**, 179–184, 227–232, 276–279 (1942); **19**, 114–118, 259–263, 515–517 (1943); **20**, 584–588 (1944); these Rev. **7**, 408, 409]. The operations are not necessarily unique and the resulting different possibilities for the definition of homomorphisms form the starting point of the theory. Under certain special conditions the ordinary isomorphism theorems are obtainable. The author discusses primitive systems and free systems and proves an analogue to the theorem of Tietze on free groups. Corresponding to group theory, embedding and extension problems are described, and also the generalization of solvable and nilpotent systems as well as representation theory. The concepts of linear and algebraic dependence are introduced and the possibility of a Galois theory examined. The section

on normal subsystems brings the decomposition theorems into relation to the general theory of Dedekind lattices.

O. Ore (New Haven, Conn.).

Thompson, Frederick B. A note on the unique factorization of abstract algebras. Bull. Amer. Math. Soc. **55**, 1137–1141 (1949).

A class of finite algebras with a single binary operation is defined. Every algebra in the class has an idempotent element, but there are simple examples of direct decompositions which are not unique up to isomorphism, and cannot be further refined. Therefore the unique factorization theorem and the refinement theorem of Jónsson and Tarski for finite algebraic systems with a $+$ and a zero cannot be extended to systems having only a $+$ and an idempotent element. *J. C. Moore* (Providence, R. I.).

Balachandran, V. K. The Chinese remainder theorem for the distributive lattice. J. Indian Math. Soc. (N.S.) **13**, 76–80 (1949).

A general form of the Chinese remainder theorem of number theory is proved by use of continued fractions. [Although the author cites a reference only for a restricted case of the theorem, it appears that his general form is also already known; see for instance C. C. MacDuffee, An Introduction to Abstract Algebra, Wiley, New York, 1940, pp. 17–18; these Rev. **2**, 241.] An analogue for distributive lattices with 0 is obtained as follows. If A is an ideal ($=\mu$-ideal$=$multiplicative ideal), say that $x\equiv y \pmod{A}$ if and only if there exist s and t in A with $x\cup s=y\cup t$. Then the system $x\equiv a \pmod{A}$, $y\equiv b \pmod{B}$ is solvable if and only if $a\equiv b \pmod{A\cup B}$; if x is a solution and $y\equiv x$ (mod $A\cap B$) then y is also a solution; any two solutions x and y satisfy $x\equiv y$ (mod $A\cap B$). The special case $A\cup B=I$ is then considered. *P. M. Whitman* (Silver Spring, Md.).

Jordan, Pascual. Über nichtkommutative Verbände. Arch. Math. **2**, 56–59 (1949).

A system with associative operations $\cap$ and $\cup$ is called a "skew-lattice" if $a\cap(b\cup a)=(a\cap b)\cup a=a$. A skew-lattice in which $(b\cup a)\cap a=a\cup(a\cap b)=a$ is a "near-lattice." A near-lattice is "modular" if

$$[(c\cup b)\cap a]\cup(b\cap c)=(c\cup b)\cap[a\cup(b\cap c)].$$

A commutative modular near-lattice is a modular lattice. Examples are given. *P. M. Whitman.*

Krishnan, V. S. Extensions of multiplicative systems and modular lattices. J. Indian Math. Soc. (N.S.) **13**, 49–59 (1949).

Suppose ϕ is a subset of the set of operations $0, I, \bigcup, \sum, \bigcap, \prod, \bigcup^*, \sum^*, \bigcap^*, \prod^*, P, S$, where * denotes that the operation is distributive in the sense of MacNeille [Trans. Amer. Math. Soc. **42**, 416–460 (1937); for other terminology see the same J. (N.S.) **11**, 49–68 (1947); **12**, 89–106 (1948); these Rev. **10**, 95, 587]. The following cases are distinguished: (1) $\phi\subset(0, I, \bigcap)$; (2) ϕ contains $\sum$ or $\prod$ and $\phi\subset(0, I, \bigcup, \sum, \bigcap, \prod)$; (3) $(0, P, \prod)\subset\phi\subset(0, I, \bigcap, \prod, P)$; (4) ϕ contains $\sum^*$ and either $\bigcap$ or $\prod$, but $\phi\subset(0, I, \bigcap, \prod, \bigcap^*, \bigcup^*, \sum^*, P)$; (5) $(\bigcap, \bigcup^*)\subset\phi\subset(0, I, \bigcap, \bigcup^*, \bigcap^*)$; (6) ϕ contains $\bigcup^*$ and either P or $\prod$, but $\phi\subset(0, I, \bigcap, \prod, \bigcap^*, \bigcup^*, P)$; (7) ϕ contains $\bigcap, \prod^*$, and either $\bigcup^*$ or $\sum^*$, but $\phi\subset(0, I, \bigcap, \bigcup^*, \sum^*, \bigcap^*, \prod^*, P)$. Theorem: A multiplicative system with units has a canonical extension to ϕ if ϕ belongs to any of the cases (1) to (5) or the dual of (3); it need not have a canonical extension to ϕ if ϕ belongs to none of the cases (1) to (7) or the dual of (3). A lattice, or modular lattice, with units has a canonical extension to ϕ if ϕ belongs to (1) to (5) or their duals; it need not have a canonical extension to ϕ if ϕ belongs to none of (1) to (7) or their duals. If ϕ belongs to (6) or (7) (or their duals in the case of a lattice or modular lattice), the problem is similarly analysed if the system is finite, but the infinite case remains unsolved. *P. M. Whitman* (Silver Spring, Md.).

Benado, Michaël. Le fondement axiomatique du théorème Jordan-Hölder relatif aux séries principales. C. R. Acad. Sci. Paris **229**, 332–334 (1949).

In the present note the author continues his investigations on the Jordan-Hölder theorem in lattices [same C. R. **228**, 529–531 (1949); these Rev. **10**, 502]. Certain normality conditions imposed by the reviewer are weakened and it is shown that also in this case an analogue of the Schreier-Zassenhaus theorem can be established.

O. Ore (New Haven, Conn.).

Dilworth, R. P. A decomposition theorem for partially ordered sets. Ann. of Math. (2) **51**, 161–166 (1950).

A subset S of a partially ordered set P is called dependent if S contains a and b with $a\geqq b$ or $b\geqq a$. The following theorems are given. If every set of $k+1$ elements of P is dependent but at least one set of k elements is not, then P is the set sum of k disjoint chains. If D is a finite distributive lattice, $k(a)$ is the number of distinct elements of D covering a, and k is the maximum of $k(a)$, then D is a sublattice of a direct union of k chains, but not of fewer. If A is any finite collection of subsets of a set, then a set T of n elements (counting repetitions) is said to represent A if there exists a one-to-one correspondence of the sets of A onto a subset of T such that each set contains its corresponding element. Suppose A and B are two finite collections of subsets (containing m and n sets, respectively) of some set; and that h is the smallest number such that, for every r, the union of any $r+h$ sets of A intersects at least r sets of B; and that k is the smallest number such that $n+k$ elements serve to represent both A and B; then $h=k$. This generalizes results of P. Hall [J. London Math. Soc. **10**, 26–30 (1935)] and Kreweras [C. R. Acad. Sci. Paris **222**, 431–432 (1946); these Rev. **7**, 376]. *P. M. Whitman.*

Dilworth, R. P., and Ward, Morgan. Note on a paper by C. E. Rickart. Bull. Amer. Math. Soc. **55**, 1141 (1949).

Simple proofs of some results of Rickart [same Bull. **54**, 758–764 (1948); these Rev. **10**, 96] on lattice isomorphisms.

P. M. Whitman (Silver Spring, Md.).

Rieger, Lad. A note on topological representations of distributive lattices. Časopis Pěst. Mat. Fys. **74**, 55–61 (1949). (English. Czech summary)

This continues investigations of Stone [same journal **67**, 1–25 (1937)]. Theorem: the space S of all prime α- (dual-) ideals of a distributive lattice L with 0 and I is a bicompact T_0-space possessing an open basis R' with the properties (i) any system of open sets of R' has a nonvoid intersection if every finite subsystem thereof does; (ii) if $Q'\subset R'$ is a system of neighborhoods of $p\varepsilon S$ with

$$\prod_{p\varepsilon A\varepsilon R'} A=\prod_{A\varepsilon Q'} A$$

and contains with A_1 and A_2 also $A_3\subset A_1\cap A_2$, then Q' is a

complete system of neighborhoods of p. Conversely, if a bicompact T_0-space S has an open basis R' satisfying (i)–(ii) then S can be taken as the space of all prime α-ideals of the distributive lattice R generated by R' with 0 and I adjoined. A bicompact T_1-space S satisfying (i)–(ii) is a totally disconnected bicompact Hausdorff space. Any distributive lattice with 0, each of whose prime α-ideals is maximal, is a generalized Boolean algebra. *P. M. Whitman.*

Terasaka, Hidetaka. Die Theorie der topologischen Verbände. Coll. Papers Fac. Sci. Osaka Univ. Ser. A. **8**, no. 1, 33 pp. (1940).

[This paper was originally to have appeared in Fund. Math. **33** (1939), and is reproduced here from a reprint. It did not appear in that volume when it was issued in 1945.] A TV (see title) is a Boolean algebra with an operation '$-$' satisfying $X^{--}=X^-$, $XX^{-\prime}=0$, $0^-=0$, $(X+Y)^-=X^-+Y^-$. It is shown that such topics as the theory of the Boolean algebra of regular open sets, of nowhere dense sets, of regular sets, etc., can be developed without reference to points, that is, for any TV. Adjoining the condition of completeness to the other TV axioms, the author continues with such topics as the identification of the interior $X'^{-\prime}$ as the least upper bound of all lesser open elements, Banach's theorem on first category sets [Kuratowski, Topologie I, 1st ed., Warszawa, 1933, p. 44] and other items treated in the concrete case in Kuratowski's book. There is an application to real-valued functions f on a topological space R. It seems that these can be embedded into a TV in such a way that f is upper (lower) semi-continuous if and only if the corresponding element is closed (open). Application of the TV theory yields Baire's theorem on the discontinuities of functions of Baire's first class. *R. Arens.*

Fuchs, Ladislas. A note on half-prime ideals. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 28, 112–114 (1948).

An ideal H in a commutative ring being called half-prime if it coincides with its radical, the author observes that H is half-prime if and only if it contains the intersection of two ideals whenever it contains their product.

I. S. Cohen (Cambridge, Mass.).

Fuchs, Ladislaus. Über die Ideale arithmetischer Ringe. Comment. Math. Helv. **23**, 334–341 (1949).

A ring R is said to be arithmetic if the distributive law $A\cap(B+C)=(A\cap B)+(A\cap C)$ holds for any three ideals. An ideal A is called primitive if $B\cap C\subset A$ implies $B\subset A$ or $C\subset A$. It is proved that R is arithmetic if and only if every irreducible ideal is primitive. If A and B are ideals in an arithmetic Noetherian ring, then there is a greatest ideal C such that $B\cap C\subset A$; it is denoted by $A\circ B$. This new operation has many of the formal properties of the ideal quotient, and it is used to prove uniqueness of the decomposition of ideals into irreducible ideals. With respect to a fixed ideal A two ideals B and C are called equivalent if $A\circ B=A\circ C$; the properties of this equivalence relation are studied.

I. S. Cohen (Cambridge, Mass.).

Fuchs, Ladislas. On primal ideals. Proc. Amer. Math. Soc. **1**, 1–6 (1950).

Un idéal A d'un anneau commutatif à élément unité R est dit primal si, dans R/A, les diviseurs de zéro forment un idéal P/A; P est un idéal premier appelé l'adjoint de A. Tout idéal primaire est primal, mais la réciproque est fausse, même si l'idéal primal est quasi-primaire (c'est-à-dire si son radical est premier). Tout idéal irréductible est primal. Par usage du théorème de Zorn, l'auteur montre que tout idéal de R est intersection des idéaux primaux qui le contiennent. Une intersection réduite (c'est-à-dire telle qu'aucune composante ne puisse être remplacée par un de ses diviseurs propres) finie d'idéaux primaux A_i est primale si, et seulement si, les idéaux premiers P_i adjoints aux A_i sont tous contenus dans l'un d'entre eux; ceci permet de définir la notion de décomposition primale "normale" d'un idéal. Dans un anneau où tout idéal est intersection finie d'idéaux irréductibles, tout idéal admet une décomposition primale normale, et le nombre de composantes de celle-ci, ainsi que leurs idéaux premiers adjoints, sont déterminés de façon unique. *P. Samuel* (Clermont-Ferrand).

Fuchs, Ladislas. Some theorems on algebraic rings. Acta Math. **81**, 285–289 (1949).

Soient F un corps de nombres algébriques, E une extension algébrique finie de degré n de F, R et P les anneaux des entiers de F et E, et P' un sous anneau de P qui soit aussi un R-module [cette dernière condition étant implicitement imposée par l'auteur]. Pour la représentation des éléments de P' comme combinaisons linéaires de $(1, x, \cdots, x^{n-1})$ (x: élément primitif de E sur F), on choisit un dénominateur commun c; les coéfficients de x^s des nombres de P' dont la représentation s'arrète à x^s forment un idéal fractionnaire $(1/c)\cdot L_s$ de F, L_s étant un idéal de R $(0\leqq s\leqq n-1)$. On a $c\epsilon L_s$ pour $s\geqq 1$, et $L_i\subset L_j$ pour $i\leqq j$. On en déduit que le discriminant relatif de P' par rapport à R est égal à $(1/c^{2n})(L_0\cdots L_{n-1})^2D(x)$, où $D(x)$ est le discriminant relatif de x [cf. Fuchs, Comment. Math. Helv. **21**, 29–43 (1948); ces Rev. **9**, 336]. *P. Samuel.*

Lesieur, Léonce. Un théorème de transfert d'un anneau abstrait à l'anneau des polynomes. Canadian J. Math. **2**, 50–65 (1950).

Consider the following property of a commutative ring A with unit: any proper ideal I admits a prime ideal divisor P such that for any $p\epsilon P$ there exist n and q not in P with $p^nq\epsilon I$. Any ring with the ascending chain condition has this property. The author's main theorem is that the property is inherited by the polynomial ring $A[x]$. The proof makes systematic use of the natural map of $A[x]$ into $K[x]$, where K is the quotient field of A/P. By similar methods he then gives a proof of the Hilbert Nullstellensatz, and of the decomposition into primary ideals in $L[x_1, \cdots, x_n]$ where L is any field; these proofs avoid the axiom of choice. [Reviewer's remark. If one is willing to use the axiom of choice, one can give a somewhat shorter proof of the main theorem, along the lines of Cohen and Seidenberg, Bull. Amer. Math. Soc. **52**, 252–261 (1946); these Rev. **7**, 410. Let J be an ideal in $A[x]$ contracting to I, and among the ideals containing J whose contraction is contained in P, select a maximal one Q; Q is prime and has the desired property relative to J.] *I. Kaplansky* (Chicago, Ill.).

Krull, Wolfgang. Parameterspezialisierung in Polynomringen. II. Das Grundpolynom. Arch. Math. **1**, 129–137 (1948).

In part I of this paper [same vol., 56–64 (1948); these Rev. **10**, 178], the author considered ideals in a polynomial ring $R=k(t)[x_1, \cdots, x_n]$ in n variables over a simple transcendental extension $k(t)$ of an infinite ground-field k, and showed that if $\mathfrak{A}(t)$ is an unmixed r-dimensional ideal in R, then $\mathfrak{A}(a)$, $a\epsilon k$, is also an r-dimensional unmixed ideal in $k[x_1, \cdots, x_n]$ almost always, i.e., but for at most a finite number of exceptions of a. In this part, he considers the

case that $\mathfrak{A}(t)$ is a prime ideal, and investigates the question whether $\mathfrak{A}(a)$ is the intersection of prime ideals almost always, giving the answer in terms of the ground-polynomial of the ideal $\mathfrak{A}$. To define this ground-polynomial, adjoin n^2 indeterminates u_{ij} to the ground-field k; the ideal $\mathfrak{A}$ extends to an unmixed r-dimensional ideal in $k(u)[x]$, which we may denote by the same symbol $\mathfrak{A}$. Let $z_i = \sum u_{ij}x_j$. In $\mathfrak{A}$ there will exist an essentially unique polynomial $F(u, z_1, \cdots, z_{r+1})$ in $z_1, \cdots, z_{r+1}$ of minimal degree in z_{r+1} which is also a polynomial in the u_{ij} and primitive in them and which is free of any factor in only r of the z's: this is the ground-polynomial of $\mathfrak{A}$. If now the ground-polynomial of the prime ideal $\mathfrak{A}(t)$ is separable, then $\mathfrak{A}(a)$ is almost always the intersection of prime ideals. The theory of the ground-polynomial for arbitrary polynomial ideals is already known, but Krull discusses the question ab initio, simplifying the theory by centering the argument about unmixed ideals. *A. Seidenberg* (Berkeley, Calif.).

Jaffard, Paul. Détermination de certains anneaux. C. R. Acad. Sci. Paris **229**, 805–806 (1949).

Employing definitions given by J. Dieudonné [Bull. Soc. Math. France **70**, 46–75 (1942); these Rev. **6**, 144] the author proves the following theorem and its converse. Every ring with non-zero right socle and such that the right annihilator of this socle is zero is contained in the direct sum of primitive rings with non-zero right socle and contains the socle of this sum. *F. Kiokemeister.*

Levi, F. W. Über den Kommutativitätsrang in einem Ringe. Math. Ann. **121**, 184–190 (1949).

The rank of commutativity of a ring R has been defined in the review of another paper of the author [J. Indian Math. Soc. (N.S.) **11**, 85–88 (1947); these Rev. **10**, 6]. The present paper is the one mentioned in the cited review as "to appear elsewhere." The rank of commutativity of a complete matric algebra of degree n over a field K is investigated. It is shown that the rank of commutativity of such an algebra depends only on n and (perhaps) on the characteristic of K. Let this rank of commutativity be $r(n)$ if the characteristic of K is 0 and $r(n, p)$ if the characteristic of K is $p \neq 0$. It is proved that $r(n)$ and $r(n, p)$ are odd, $r(2) = r(2, p) = 3$, $r(3) = r(3, p) = 5$, and in general $2n-1 \leqq r(n, p) \leqq r(n) < n^2$, $r(n)+2 \leqq r(n+1)$, $r(n, p)+2 \leqq r(n+1, p)$. The determination of $r(n)$ in general, and the question of whether $r(n, p) = r(n)$ for $3 < n$, appear to be difficult. The author's proofs are of intrinsic interest, since they involve a consideration of some of the properties of the noncommutative "determinant" $[a_1, a_2, \cdots, a_k]$. An application of the function $r(n)$ may be found in a paper of Kaplansky [Canadian J. Math. **1**, 105–112 (1949); these Rev. **10**, 428]. *S. A. Jennings* (Vancouver, B. C.).

McCoy, Neal H. Prime ideals in general rings. Amer. J. Math. **71**, 823–833 (1949).

An ideal $\mathfrak{p}$ of a ring R is shown to be prime if and only if $aRb \equiv 0\ (\mathfrak{p})$ implies $a \equiv 0\ (\mathfrak{p})$ or $b \equiv 0\ (\mathfrak{p})$. A set $M \subset R$ is called an m-system if $a, b \in M$ implies $axb \in M$ for some $x \in R$. The radical $\mathfrak{r}$ of an ideal $\mathfrak{a}$ consists of all $r \in R$ not belonging to an m-system disjoint from $\mathfrak{a}$. It is proved that $\mathfrak{r}$ is the intersection of all prime ideals containing $\mathfrak{a}$. The radical N of R is defined to be the radical of the zero ideal. Then N is a nil ideal that coincides with the classical radical in the presence of the descending chain condition on right ideals. Many of the expected properties of a radical follow. If the zero ideal is prime in R, R is called a prime ring. A ring has zero radical if and only if it is a subdirect sum of prime rings. *R. E. Johnson* (Northampton, Mass.).

Smiley, Malcolm F. Application of a radical of Brown and McCoy to non-associative rings. Amer. J. Math. **72**, 93–100 (1950).

In this paper the author points out that the theory of "radicals" of an associative ring due to Brown and McCoy [same J. **69**, 46–58 (1947); these Rev. **8**, 433] applies without alteration to nonassociative rings (narings). A particular Brown-McCoy radical, the F_1-radical, is then considered. This is defined as follows. If R is a naring and a is any element of R, $F_1(a)$ denotes the ideal of R generated by the set of all elements of the form $ax-x+ya-y$, x and y being arbitrary elements of R. The F_1-radical N of R then consists of all elements b of R such that every element a of the ideal generated by b satisfies the condition $a \in F_1(a)$. Then N is an ideal of R and $R-N$ has zero F_1-radical. The author then proves that a simple naring is semi-simple (i.e., has zero F_1-radical) if and only if it has a unit element, and that, further, semi-simple narings are sub-direct sums of simple narings with unit element. If the descending chain condition holds for the ideals of a semi-simple naring R, then "sub-direct" can be replaced by "direct." Finally the author proves two results relating this definition of the radical to other definitions. The first is that, for a nonassociative algebra with a unit, the F_1-radical coincides with the radical defined by Albert [Bull. Amer. Math. Soc. **48**, 891–897 (1942); these Rev. **4**, 130], while the second is that, for alternative rings satisfying certain chain conditions, it coincides with the radical defined by Zorn [Ann. of Math. (2) **42**, 676–686 (1941); these Rev. **3**, 100]. *D. Rees* (Cambridge, England).

Motzkin, Th. The Euclidean algorithm. Bull. Amer. Math. Soc. **55**, 1142–1146 (1949).

The author obtains a constructive criterion for the existence of a Euclidean algorithm within an integral domain Q. The derived set S' of any subset S of Q is defined as the set of all $b \in S$ for which there exists an $a \in Q$ such that $a+bQ \subseteq S$. Then Q has a Euclidean algorithm if and only if the intersection $\bigcap P_0^{(\iota)}$ of the sequence

$$Q-0 = P_0 \supseteq P_0' \supseteq P_0'' \supseteq \cdots \supseteq P_0^{(\iota)} \supseteq \cdots$$

is empty. This criterion is extended to the case of a "transfinite" algorithm, in which the norm $|b|$ is allowed to take any ordinal numbers as values. The criterion is applied to some special rings, and an example is given to show the existence of principal ideal rings with no (transfinite) algorithm. Different sets of axioms for the Euclidean algorithm and related notions are compared, and possible implications for the classification of principal ideal rings and other integral domains are indicated. *B. N. Moyls.*

Neumann, B. H. On ordered division rings. Trans. Amer. Math. Soc. **66**, 202–252 (1949).

Die Arbeit zerfällt in zwei unabhängige Teile. Im 1. Teil wird bewiesen, dass sich jede geordnete Gruppe in einen geordneten Divisionsring einbetten lässt. Da sich die freie Gruppe mit beliebig vielen Erzeugenden anordnen lässt, folgt insbesondere, dass der Gruppenring der freien Gruppe sich in einen geordneten Divisionsring einbetten lässt, womit ein vom Ref. gestelltes Problem [J. Reine Angew. Math. **176**, 203–223 (1937)] gelöst ist. Das Kriterium von Ore ist jedoch in diesem Falle nicht erfüllt, denn im Gruppenring

der freien Gruppe aus zwei Erzeugenden lassen sich zwei Elemente ohne gemeinsames Linksvielfache angeben. Der Beweis des Haupttheorems verallgemeinert Hahn's Ideen [Akad. Wiss. Wien, S.-B. IIa. **116**, 601–655 (1907)] auf den nichtkommutativen Fall. Verfasser zeigt, dass der Ring der formalen Potenzreihen einer geordneten Gruppe über einem Divisionsring selbst ein Divisionsring ist, der überdies geordnet werden kann, wenn der Divisionsring der Koeffizienten anordbar ist. Der Kernpunkt ist der Nachweis der Existenz der Inversen zu jedem von Null verschiedenen Element des Potenzreihenrings. Es genügt, die Inverse zu einer Potenzreihe aufzuzeigen, die mit $e \cdot 1$ beginnt (e Gruppeneinheit, 1 Koeffizienteneinheit). Die Elemente g aus G mit $g>e$ bilden eine Semigruppe G^*, mit der zunächst operiert wird. Die Elemente $r, s, t, \cdots$ von G^* sind linear geordnet und genügen den Bedingungen: aus $r<s$ folgt $tr<ts$ und $rt<st$; es ist stets $r<r^2$. Sie lassen sich in Klassen von Elementen einteilen, die je archimedisch miteinander vergleichbar sind (d.h. bei zwei Elementen derselben Klasse ist nicht jede Potenz eines Elementes grösser bzw. kleiner als das andere Element). Dies ermöglicht die Einführung des Begriffes einer wohlgeordneten Reihe (w.o. Reihe) $S \leqq G^*$, bei der jede nicht leere Untermenge ein kleinstes Element besitzt. Produkt und Vereinigungsmenge endlich vieler w.o. Reihen ist wieder eine w.o. Reihe. Unter ihnen ist die w.o. Reihe S^ω als Vereinigungsmenge aller S^n ($n=1, 2, 3, \cdots$) besonders wichtig; jedes Element von S^ω kommt nur in endlich vielen der Mengen $S, S^2, S^3, \cdots$ vor. Sei P ein Ring mit Operatorenbereich Ω, die Elemente ∞ von P werden als Exponenten geschrieben. Eine Funktion φ, die über G^* erklärt ist und Werte aus P hat, wird als eine formale Potenzreihe bezeichnet, falls eine w.o. Reihe $S(\varphi)$ aus G^* existiert so, dass aus $\varphi(s) \neq 0$ folgt $s \varepsilon S(\varphi)$. Zwischen diesen Potenzreihen ist die Addition und die skalare Multiplikation wie üblich erklärt. Die Multiplikation zweier formaler Potenzreihen wird erklärt mit Hilfe von Faktorpaaren $\gamma(r, s)$ und $\omega(r)$, wo $\gamma(r, s)$ und $\omega(r)$ für alle Elemente r, s aus G^* erklärt its und $\gamma(r, s)$ in P, $\omega(r)$ in Ω liegt und identisch in r, s, t aus G^* und ρ aus P gilt:

$$\gamma(r, s)\rho^{\omega(r)\omega(s)} = \rho^{\omega(rs)}\gamma(r, s),$$
$$\gamma(rs, t)\gamma(r, s)^{\omega(t)} = \gamma(r, st)\gamma(s, t).$$

Das Produkt $\varphi\chi$ zweier formaler Potenzreihen ist dann die formale Potenzreihe ψ mit

$$\psi(t) = \sum_{rs=t} \gamma(r, s)\varphi(r)^{\omega(s)}\chi(s).$$

Diese Multiplikation ist assoziativ und distributiv. Die Menge aller formalen Potenzreihen bildet einen Ring Π ohne Einselement, der additiv und multiplikativ abgeschlossen ist. Die Symbole $1+\varphi$ für alle φ aus Π bilden dann bei der Komposition

$$(1+\varphi)(1+\chi) = 1+(\varphi+\chi+\varphi\chi)$$

eine multiplikative Gruppe Γ. Die Einbettung einer geordneten Gruppe G mit Einselement in einen Divisionsring gelingt jetzt dadurch, dass jeder nicht verschwindenden formalen Potenzreihe über G mit Hilfe eines Elementes aus Γ dargestellt werden kann in der Form

$$\phi = \pi_r \rho(1+\varphi), \qquad r \varepsilon G,\ \rho \varepsilon P,\ 1+\varphi \varepsilon \Gamma.$$

Dabei ist π_r die spezielle Potenzreihe mit $\pi_r(r)=1$, $\pi_r(s)=0$, falls $r \neq s$; π_1 ist das Einselement der Multiplikation. Ist P speziell geordnet, so wird der Divisionsring der Elemente ϕ geordnet durch die Festsetzung $\phi>0$, falls $\rho>0$.

Im 2. Teil der Arbeit wird der Satz, dass jeder geordnete Körper so erweitert werden kann, dass er den Körper der reellen Zahlen enthält, durch Bereitstellung umfangreicher Hilfsbetrachtungen auf den nicht-kommutativen Fall übertragen: jeder geordnete Divisionsring Σ kann erweitert werden zu einem geordneten Divisionsring Σ^*, der Σ in seiner alten Anordnung enthält und in seinem Zentrum einen Körper, der zum Körper aller reellen Zahlen isomorph ist. Die Erweiterung von Σ erfordert die Adjunktion algebraischer und transcendenter Elemente θ und wird nach dem Vorbild von Steinitz [Algebraische Theorie der Körper, de Gruyter, Berlin, 1930] in eine Kette einfacher Erweiterungen aufgelöst. Der Nachweis, dass jede solche Adjunktion von θ möglich ist, d.h., dass $\Sigma(\theta)$ wieder ein Divisionsring ist, ist leicht. Schwierig ist die Lösung der weiteren Aufgabe, $\Sigma(\theta)$ so zu ordnen, dass die ursprüngliche Ordnung innerhalb Σ erhalten bleibt und gleichzeitig θ den richtigen Anordnungsbeziehungen unterworfen wird bezüglich der "rationalen" Elemente $\rho \varepsilon \Sigma$, wie sie von dem reellen Bild ${}^*\theta$ bezüglich der gewöhnlichen rationalen Bilder ${}^*\rho$ erfüllt werden, d.h. dass aus ${}^*\rho_1 < {}^*\theta < {}^*\rho_2$ stets folgt $\rho_1 < \theta < \rho_2$ wo die ${}^*\rho_i$ dem Primkörper der rationalen Zahlen und ρ_i dem Primkörper P_0 von Σ angehören und zwischen P_0 und den rationalen Zahlen ein anordnungstreuer Isomorphismus besteht. Die Elemente von Σ zerfallen in den Bereich $\mathfrak{R}$ der endlichen Elemente σ, die nicht kleiner bzw. grösser sind als alle Elemente ρ_0 aus P_0; den Bereich $\mathfrak{P}$ der Elemente, die nichtarchimedisch klein sind verglichen mit dem Einselement des Ringes und den Bereich der unendlichen Elemente, die grösser bzw. kleiner sind als alle Elemente aus P_0. Es sei P beliebiger Unterkörper des Durchschnitts von $\mathfrak{R}$ und dem Zentrum von Σ. Ein endliches Element σ definiert in Σ einen Dedekindschen Schnitt und daher vermöge des Anordnungsisomorphismus von P_0 auf ${}^*P_0$ (${}^*P_0$ Körper der rationalen Zahlen) einen Schnitt in ${}^*P_0$, der eine eindeutige reelle Zahl ${}^*\sigma$ definiert. Die Elemente von $\mathfrak{R}$ werden so auf den reellen Teilbereich ${}^*\mathfrak{R}$ abgebildet. Die Anordnung einer einfachen transcendenten Erweiterung von Σ bietet keine Schwierigkeiten. Für die Anordnung algebraischer Erweiterungen gilt dies auch, falls das zu adjungierende Element nicht unendlich benachbart ist zu schon vorhandenen Elementen des Divisionsrings. Dagegen fordert die Anordnung von algebraischen Erweiterungen durch Elemente, die zu bereits vorhandenen Elementen unendlich benachbart sind, weitläufige Untersuchungen. Um eine solche Erweiterung durch θ mit ${}^*\theta \varepsilon {}^*\mathfrak{R}$ anzuordnen, spielen die zu θ unendlich benachbarten Zahlen ξ aus $\mathfrak{R}$ eine besondere Rolle, die nicht unendlich klein sind gegen 1, die das Bild der irreduziblen Gleichung von θ mit Koeffizienten aus ${}^*P$ bis auf einen unendlich kleinen Fehler erfüllen und bei der Abbildung von P auf ${}^*P$ in ${}^*\theta$ übergehen. In ihnen hat ein beliebiges Polynom p mit endlichen und nicht sämtlich zu kleinen Koeffizienten und kleinerem Grad als die irreduzible Gleichung für θ ein bestimmtes Vorzeichen; ein Polynom, das in θ nicht verschwindet, hat also in einer gewissen, genauer zu definierenden Nähe von θ konstantes Vorzeichen. Unter den Zahlen ξ gibt es solche, für die $f(\xi)/p(\xi)$ eine in $\mathfrak{P}$ gelegene Zahl ist. Dann setze man fest: $p(\theta) \gtrless 0$ in $\Sigma(\theta)$, je nachdem $p(\xi) \gtrless 0$ in Σ. Darauf lässt sich dann eine Anordnung aller Polynome des Polynombereiches $\Sigma[x]$ einführen. Der Nachweis der Existenz der Hilfsgrössen erfordert längere vorbereitende Überlegungen, die zum Teil denen von W. Wagner [Math. Ann. **113**, 528–567 (1936)] verwandt sind. *R. Moufang* (Frankfurt am Main).

Amitsur, Shimshon. On a lemma of Kaplansky. Riveon Lematematika 3, 47–48, 52 (1949). (Hebrew. English summary)

Let F be a field containing m ($\leqq\infty$) elements and let A be an algebra over F satisfying an identity $g(a_1, \cdots, a_n)=0$, $a_i \epsilon A$, where $g(x_1, \cdots, x_n)$ is in the free polynomial ring $F[x_1, \cdots, x_n]$ and is of degree less than m in each x_i. It is proved that if K is a field containing F, then $g=0$ holds also in $A\times K$ (Kronecker product). The case where g is linear in each argument is Kaplansky's lemma [Bull. Amer. Math. Soc. 54, 575–580 (1948); these Rev. 10, 7]. An example is given to show that the restriction on the degree cannot be removed. The proof may be outlined as follows. Define $p(g)=\sum t^h\mu_h$, where t is an indeterminate and μ_h is the number of x_i in g having degree h. The proof proceeds by induction on $p(g)$, these being well ordered by $p(g_1)>p(g_2)$ if $p(g_1)-p(g_2)$ has positive leading coefficient. If

$$q(u, v, x_2, \cdots, x_n)=g(u+v, x_2, \cdots, x_n) -g(u, x_2, \cdots, x_n)-g(v, x_2, \cdots, x_n),$$

then A clearly satisfies $q=0$, and since $p(q)<p(g)$, $A\times K$ satisfies $q=0$ by the induction assumption. This means that, over $A\times K$, g is additive in x_1, likewise in each x_i. Hence it suffices to prove $g(a_1k_1, \cdots, a_nk_n)=0$ for $a_i\epsilon A$, $k_i\epsilon K$. Write $g=\sum g_j$, where g_j is homogeneous in each x_i. Kaplansky proved that A must satisfy each $g_j=0$, hence $g_j(a_1k_1, \cdots, a_nk_n)=0$, q.e.d. *I. S. Cohen.*

Mann, H. B. On the field of origin of an ideal. Canadian J. Math. 2, 16–21 (1950).

The ideals considered are integral ideals in finite algebraic extensions of the rational number field. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are considered to be equal if their elements generate the same ideal in a field containing all the elements of both $\mathfrak{a}$ and $\mathfrak{b}$. It is therefore possible to speak of an ideal without specifying any particular field in which it lies. Henceforth only fields are considered which are finite algebraic extensions of a fixed field $\mathfrak{F}$, itself a finite algebraic extension of the rational field; and an admissible subfield is one containing $\mathfrak{F}$. The ideal $\mathfrak{a}$ is said to originate in $\mathfrak{F}_1$ over $\mathfrak{F}$ if $\mathfrak{a}$ is contained in $\mathfrak{F}_1$ but not in any proper admissible subfield of $\mathfrak{F}_1$. If $\mathfrak{F}_1\supset\mathfrak{F}_2$ and $\mathfrak{a}$ is an ideal in $\mathfrak{F}_1$, the elements of $\mathfrak{a}$ which lie in $\mathfrak{F}_2$ form an ideal $\mathfrak{A}$ in $\mathfrak{F}_2$ which is said to correspond to $\mathfrak{a}$. If $\mathfrak{A}\subset\mathfrak{F}$ corresponds to $\mathfrak{a}$ in $\mathfrak{F}_1$ and $\mathfrak{A}=\mathfrak{a}^e\mathfrak{c}$ where $(\mathfrak{a}, \mathfrak{c})=1$, then $\mathfrak{a}$ is said to be of order e with respect to $\mathfrak{F}$. If $\mathfrak{a}$ is an ideal of order 1 with respect to $\mathfrak{F}$, then $\mathfrak{a}$ originates in a unique subfield $\mathfrak{F}_1$ over $\mathfrak{F}$, and an extension of $\mathfrak{F}$ contains $\mathfrak{a}$ if and only if it contains $\mathfrak{F}_1$. If $\mathfrak{p}$ is an ideal in a field over $\mathfrak{F}$ and g is the largest integer for which $\mathfrak{p}^g$ is a prime ideal in some extension of $\mathfrak{F}$, then $\mathfrak{p}^g$ originates in a unique extension $\mathfrak{F}'$ over $\mathfrak{F}$, and is a prime ideal in $\mathfrak{F}'$. If $\mathfrak{p}^g$ $(g>1)$ is prime in some extension of $\mathfrak{F}$, there are an infinite number of extensions of $\mathfrak{F}$ in which $\mathfrak{p}$ originates and is prime. Several other related results are also obtained.

N. H. McCoy (Northampton, Mass.).

Pickert, G. Inseparable Körpererweiterungen. Math. Z. 52, 81–136 (1949).

This is a systematic exposition of the whole theory and, being self contained, gives proofs of both old and new results. The main tools used are the concepts of exponent of an inseparable extension, p-independence and degree of imperfection [Teichmüller, Deutsche Math. 1, 362–388 (1936)], multiplicity (minimum number of generators), the Kronecker product, $M\times L$, over K, of two extension fields of K, and a normal form $L=K(a_1, \cdots, a_m)$ for pure inseparable L/K, introduced earlier by the author [Math. Ann. 116, 217–280 (1938); Ber. Math.-Tagung Tübingen 1946, 118–120 (1947); these Rev. 9, 77] in which each a_i has with respect to $K(a_1, \cdots, a_{i-1})$ a nonzero exponent e_i, with $e_1\leqq\cdots\leqq e_m$. These e_i are invariants of L/k. The first section studies the behavior of the e_i, and m, under extension of the ground field. The second studies the relation between an extension field L/R and the prime divisors in the ground field. An example shows that there is no simple relation between the multiplicities of L/K and of the corresponding residue class fields, even when L is generated by an extension of the field of constants. An error in the first paper of the author cited above is corrected. The third section studies infinite extensions. Results stated by the author in the second of his papers cited above are proved and extended. *G. Whaples* (Bloomington, Ind.).

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Hasse, Helmut. Die Multiplikationsgruppe der Abelschen Körper mit fester Galoisgruppe. Abh. Math. Sem. Univ. Hamburg 16, 29–40 (1949).

This paper deals with a generalization of Albert's [Structure of Algebras, Amer. Math. Soc. Colloquium Publ., v. 24, New York, 1939, chapter VI; these Rev. 1, 99] and Teichmüller's [Deutsche Math. 1, 197–238 (1936)] theory of cyclic semi-fields. An Abelian semi-field K/Ω over the field Ω is a separable commutative semi-simple algebra with an Abelian group of automorphisms $\mathfrak{A}$ such that K is operator isomorphic with respect to the group ring $\mathfrak{A}_\Omega$ for a right operator set to $\mathfrak{A}_\Omega$ as a right ideal. This condition implies that a basis of K/Ω is transformed by each automorphism A of $\mathfrak{A}$ by means of a linear transformation which belongs to the class of the regular representation of $\mathfrak{A}$ in Ω. If the order $[\mathfrak{A}:1]$ is relatively prime to the characteristic of Ω and if Ω contains all $[\mathfrak{A}:1]$-th roots of unity, then the algebra K/Ω has special bases $\{\omega_\chi\}$, so-called factor bases, with respect to which the matrix representing a given automorphism A is diagonal and shows the characters $\chi(A)$, i.e., $\omega_\chi^A=\chi(A)\omega_\chi$ with regular elements ω_χ in K. Expressing the effect of an element A upon $\omega_\chi\omega_\psi$ and $\omega_{\chi\psi}$ a factor set $\mathfrak{c}_{\chi,\psi}$ of the character group X of A in the multiplicative group Ω^* is obtained. Passage to other factor bases shows that the class $\mathfrak{c}$ of associated factor sets $\mathfrak{c}_{\chi,\psi}$ of K/Ω is of special significance. The author discusses furthermore how an Abelian semi-field can conversely be constructed from a given class of factor sets with the relations $\mathfrak{c}_{\chi,\psi}=\mathfrak{c}_{\psi,\chi}$ and $\mathfrak{c}_{\varphi\chi,\psi}\mathfrak{c}_{\varphi,\chi}=\mathfrak{c}_{\varphi,\chi\psi}\mathfrak{c}_{\chi,\psi}$. In this construction a typical component of K/Ω is determined by means of the theory of radical extensions. A multiplication of semi-fields K_i/Ω is defined after isomorphisms π_i of the automorphism groups $\mathfrak{A}_i$ of the algebras K_i upon the same abstract group $\mathfrak{A}$ are fixed once and for all. Thus the pairs $(K_i/\Omega, \pi_i)$ are taken and in their direct product (in the sense of the theory of algebras) the semi-field $(K_0/\Omega, \pi_0)$ is picked out as the algebra of the elements which are invariant for the elements $\prod_i A_i$ in the direct product $\prod_i\mathfrak{A}_i$ for which $\prod_i A_i^{\pi_i}=1$ in $\mathfrak{A}$. A discussion of the effect of changing the isomorphisms π_i by automorphisms of $\mathfrak{A}$ leads to the equality relation $(K'/\Omega, \pi')=(K/\Omega, \pi)$ if and only if $\mathfrak{c}'=\mathfrak{c}$ for the corresponding classes of factor sets. A direct verification shows that $\mathfrak{c}_0=\prod_i\mathfrak{c}_i$ for the corresponding classes with fixed isomorphisms π_i, π_0. Consequently the semi-fields $(K/\Omega, \pi)$ (noting that fixed selection is made for all π), form a group which is isomorphic to the group of classes of factor sets $\mathfrak{c}=\{\mathfrak{c}_{\varphi,\psi}\}$, subject to the above associativity and commutativity relations, of the character group of $\mathfrak{A}$ in Ω^*. Finally,

the author rephrases the determination of the product $(K_0/\Omega, \pi_0)$ by means of the class field theory (Artin symbol and norm residue symbol) for special ground fields; e.g., the field component of $(K_0/\Omega, \pi_0)$ is determined by the class group H_0 of idèles $\mathfrak{a}$ of Ω which are given by the requirement

$$\prod_i \left(\frac{K_i/\Omega}{\mathfrak{a}}\right)^{r_i} = 1,$$

where ( ) denotes the Artin symbol belonging to a typical field component of the algebra $(K_i/\Omega, \pi_i)$.

O. F. G. Schilling (Chicago, Ill.).

Kochendörffer, Rudolf. Bemerkung zu einer Arbeit von H. Hasse. Math. Nachr. **2**, 245–250 (1949).

It had been shown by the reviewer [J. Reine Angew. Math. **168**, 44–64 (1932)] that certain imbedding problems for fields are equivalent to problems of the theory of algebras. The author uses the methods of H. Hasse [Math. Nachr. **1**, 40–61, 213–217, 277–283 (1948); these Rev. **10**, 426, 503] to give a new proof for these results. A number of remarks are added. *R. Brauer* (Ann Arbor, Mich.).

Skolem, Th. Two generalizations of a well known theorem on polynomials. I. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 19, 70–73 (1948).

It is known that unique decomposition into prime elements is passed on from a domain of integrity to its polynomial domain. This is generalized here for a multiplicative system O which is the set theoretic sum of a set of additive Abelian groups $\mathfrak{o}_r$ which are in one-to-one correspondence with the elements r of a group G. Any two of them, $\mathfrak{o}_r$ and $\mathfrak{o}_s$, have only the zero-element in common. The elements of $\mathfrak{o}_r$ and $\mathfrak{o}_s$ can be multiplied in such a way that $\mathfrak{o}_r\mathfrak{o}_s \subset \mathfrak{o}_t$ if $rs=t$. If e is the unit element of G then $\mathfrak{o}_e$ is supposed to contain the multiplicative unit 1. Let then $\mathfrak{o}_r(x)$ be the set of polynomials in x with coefficient in $\mathfrak{o}_r$ and $O(x)$ the set theoretic sum of the $\mathfrak{o}_r(x)$. If O has unique decomposition into primes so has $O(x)$. *O. Todd-Taussky* (Washington, D. C.).

Skolem, Th. Two generalizations of a well known theorem on polynomials. II. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 20, 74–77 (1948).

The author gives another generalization of the theorem mentioned in the review above. Consider expressions $\frac{\mathfrak{a}}{\mathfrak{b}} f(x)$ when $\mathfrak{a}$, $\mathfrak{b}$ are integral ideals in an algebraic number field F and $f(x)$ a polynomial with coefficients in F. Assume that the product of $\frac{\mathfrak{a}}{\mathfrak{b}}$ and a coefficient of $f(x)$ is always an integral ideal. The value of $\frac{\mathfrak{a}}{\mathfrak{b}} f(x)$ if x is replaced by an integral number of F is then an integral ideal, which is here called an i-polynomial. These are shown to form a multiplicative semi-group G with a single unit element, namely (1) and left cancellation law. An element of G is called prime, if it can only be expressed as a product of two elements when one of them is the unit. It is shown that the elements other than (1) are unique products of primes.

O. Todd-Taussky (Washington, D. C.).

Deuring, Max. Algebraische Begründung der komplexen Multiplikation. Abh. Math. Sem. Univ. Hamburg **16**, 32–47 (1949).

The theory of complex multiplication, i.e., the description of the Abelian extensions of an imaginary quadratic field Σ by means of special (singular) values of the modular invraiant $j(z)$ and Weber's elliptic τ-function, is based upon identities and congruences between the Fourier coefficients of $j(z)$ at infinity and rather explicit formulas for the multiplication of the τ-function. In the classical treatment the use of the modular group and the transformation groups modulo an integer is essential; such an approach has no obvious analogue in the theory of moduli for abstract function fields. In this paper a more algebraic method of presentation is described and Weber's definition of a class field is emphasized. Thus the primary aim is to obtain an algebraic proof of the decomposition law for almost all prime ideals of Σ. The author leans heavily on his previous work on the moduli of (abstract) elliptic function fields [Abh. Math. Sem. Hansischen Univ. **14**, 197–272 (1941); Math. Z. **47**, 47–56 (1940); same Abh. **15**, 211–261 (1947); these Rev. **3**, 104, 266; **10**, 5] and the reduction of coefficient fields modulo discrete rank one valuations [Math. Z. **47**, 643–654 (1942); these Rev. **7**, 362]. The procedure for the absolute class field of a given field Σ with the maximal order of integers $\mathbf{R}$ is about the following. The analytic theory of elliptic functions furnishes the existence of an elliptic field $\hat{K}=\hat{k}(x, y)$ over the complex number field $\hat{k}$ with $\mathbf{R}$ for the multiplication ring. Then algebraic reasoning may be used in order to show that $\hat{K}$ is characterized, to within isomorphisms over $\hat{k}$, by a complex number j, the module of $\hat{K}$. This number is in the classical case the value of $j(z)$ for a basis of $\mathbf{R}$. This value j is adjoined to the rational number field so as to obtain the smallest field of definition for $\hat{K}$ over the rational number field. Similarly any abstract elliptic field is determined by its module in a strictly algebraic manner; the abstract modules are algebraic over the respective prime fields, a result which was proved by the author by an algebraization of the classical reasoning based upon Fourier expansions. Next each ideal $\mathbf{a}$ of $\mathbf{R}$ gives rise to an elliptic subfield $\hat{K}^{\mathbf{a}}=\bigcup_{\mu\in\mathbf{a}}\hat{K}^{\mu}$, where $\hat{K}^{\mu}$ is defined algebraically as a subfield of $\hat{K}$, which is isomorphic to $\hat{K}$. The field $\hat{K}^{\mathbf{a}}$ has an invariant $j(\mathbf{k})$, where $\mathbf{k}$ is the absolute ideal class of $\mathbf{a}$, and where the independence of $\mathbf{a}$ as a representative of $\mathbf{k}$ is established algebraically. (Note that this kind of independence is proved in the classical theory by means of the invariantive properties of $j(z)$ with respect to the modular group.) Furthermore an algebraic proof is given to show that all fields $\Sigma(j(\mathbf{k}))$, for variable $\mathbf{k}$, are equal to $\Omega=\Sigma(j)$. Next the field $\Omega(x, y)=K$ with $K\hat{k}=\hat{K}$ is considered and a prime ideal P of Ω with contraction p to Σ is used for the reduction of the elements in Ω to $\bar{\Omega}$ and of K to a field $\bar{K}$. For almost all primes P the corresponding fields $\bar{K}$ are elliptic and their corresponding multiplication rings contain isomorphic images of the given ring $\mathbf{R}$. Moreover $\bar{K}$ is inseparable over $\overline{K^{\mu}}$ if and only if μ lies in the ideal generated by p. If p has the absolute degree 1, then the abstract theory of multiplication implies $\overline{K^{\mathbf{a}p}}=(\overline{K^{\mathbf{a}}})^p$ for all ideals $\mathbf{a}$. Repeated application of the algebraic theory of moduli yields the basic congruence $j(\mathbf{k}p)\equiv j(\mathbf{k})^p \pmod{p}$ for all but a finite number of primes, for the residues of the moduli are the moduli of the reduced elliptic fields. Thus the correct decomposition law holds for Ω/Σ and Weber's definition implies that Ω is the absolute class field of Σ. Finally the author treats the (more complicated) theory of ray class fields by means of further refinements of the procedure indicated above. *O. F. G. Schilling* (Chicago, Ill.).

Herstein, Israel Nathan. Divisor algebras. Amer. J. Math. 71, 800–822 (1949).

Soit K un corps de fonctions algébriques d'une variable avec le corps de constantes k algébriquement clos. L'ensemble de ses diviseurs est un groupe multiplicatif abélien G, ayant un sous-groupe H, celui des diviseurs principaux, qui peut être considéré comme un espace projectif désarguien sur k: en effet, si (f) est le diviseur principal défini par un $f\epsilon K$, on a $(f)=(g)$ si, et seulement si, $fg^{-1}\epsilon k$, et, K étant considéré comme un espace affine sur k, H est, visiblement, l'espace projectif correspondant. Sa dimension est >1, car $(K:k)>2$. Si l'on définit la structure d'espace projectif de H par la seule relation de dépendance linéaire, on constate que si une projectivité conserve 3 points distincts d'une droite de H, elle conserve tout point de cette droite. D'autre part la multiplication par un $C\epsilon H$ conserve la dépendance ou l'indépendance des diviseurs. Il en résulte que toute classe dans G (mod H) s'organise en un espace projectif isomorphe au précédent, si l'on définit ses sous-espaces linéaires comme produits de ceux de H par un diviseur fixe appartenant à la classe (ces sous-espaces ne dépendant pas du choix de ce diviseur). Ainsi, l'ensemble G/H des classes dans G (mod H) s'organise en une famille d'espaces projectifs isomorphes. On dira qu'on a une algèbre faible de diviseurs quand on a un groupe abélien $\bar{G}$, où est donné un sous-groupe $\bar{H}$, organisé en un espace projectif par la relation de dépendance linéaire, les relations de multiplication dans $\bar{G}$ et de l'indépendance linéaire dans $\bar{H}$ satisfaisant aux conditions qu'on vient d'énumérer; alors, $\bar{G}/\bar{H}$ peut s'organiser en une famille d'espaces projectifs isomorphes. Les diviseurs de K possèdent encore d'autres propriétés: le groupe abélien G est libre, et parmi ses bases on peut distinguer celle que forment les diviseurs premiers de K. La donnée de ces diviseurs permet de définir, d'une manière habituelle, l'intégrité d'un diviseur et la divisibilité d'un diviseur par un autre. On constate que: (α), tout diviseur dépendant linéairement de diviseurs entiers l'est aussi, et (β), pour tout diviseur premier P, il existe, sur toute droite de chacun des espaces projectifs de la famille G/H, au moins un point divisible par P.

Si le groupe $\bar{G}$ d'une algèbre faible de diviseurs est libre, et si l'on s'y donne une base, l'algèbre faible dont la structure est complétée par la relation d'appartenance à cette base, sera dite une algèbre de diviseurs si la divisibilité dans $\bar{G}$, définie à partir de cette base, satisfait aux conditions indiquées. En particulier, l'ensemble des diviseurs de K, où la multiplication, la dépendance linéaire et la divisibilité sont pris avec leur sens habituel, sera dit l'algèbre de diviseurs de K/k. Dans une algèbre des diviseurs de K/k, k est l'ensemble des rapports anharmoniques $a(A, B, C, D)$ des quadruples colinéaires de points de H (ou de toute autre espace de la famille G/H). Or, l'égalité des rapports anharmoniques et les opérations rationnelles avec ces rapports peuvent se définir, d'une manière purement géométrique, à partir de la seule relation de colinéarité, par les constructions bien connues. Ces constructions peuvent se faire dans tout espace projectif abstrait de dimension >1 où toute projectivité conservant 3 points colinéaires, conserve tout point de leur droite. On obtient ainsi un corps $\bar{k}$, qui est commutatif si l'espace considéré est désarguien. A, B, C étant 3 diviseurs colinéaires, il existe un et un seul $f\epsilon K$ tel que $(f)=A/C$ et $(f-1)=B/C$; f_{ABC} étant cet élément de K, on a $f_{ABC}=f_{DEF}$ si et seulement si les triples A, B, C, D, E, F sont proportionnels dans G. L'auteur montre qu'on peut construire, en ne se servant que de la multiplication et de la colinéarité dans G, les triples de diviseurs colinéaires tels que les $f\epsilon K$ correspondants soient les résultats des 4 opérations rationnels faites sur les éléments donnés f_{ABC} et f_{EDF} de K où les produits et les sommes de f_{ABC} avec un $\lambda\epsilon k$. Ainsi, par exemple, $f_{ABC}+f_{DEF}=f_R, S, CF$, où S est le point d'intersection de droites (CD, BF) et (CE, AF), et où R est celui de (CF, S) et de (CD, AF). Cette construction est toujours réalisable, dans une algèbre faible de diviseurs, et y donne un corps commutatif. Ainsi, dans une algèbre faible de diviseurs, il est toujours possible de définir 2 corps $\bar{k}$ et $\bar{K}\supset\bar{k}$ qui coïncident avec k et K quand il s'agit de l'algèbre faible des diviseurs de K/k. Si A, B sont linéairement indépendants, pour tout n, $A^n, A^{n-1}B, \cdots, AB^{n-1}, B^n$ le sont aussi, car, en supposant les résultat prouvé pour $n-1$ et en prenant A, B premiers entre eux, B^n ne peut dépendre des $A^n, A^{n-1}B, \cdots, AB^{n-1}$ que si B^n/A (qui dépend des diviseurs entiers $A^{n-1}, A^{n-2}B, \cdots, B^{n-1}$) est entier. Ce raisonnement est valable pour toute algèbre de diviseurs et montre que tout $f\epsilon\bar{K}$ non constant est transcendant par rapport à $\bar{k}$.

Tout $f\epsilon K$ peut être considéré comme une fonction $f(P)$ définie pour tout diviseur premier P et à valeurs dans k, en posant $f(P)$ égal au reste constant de f (mod P), et $f\to f(P)$ est un isomorphisme par rapport aux opérations rationnelles; or, l'algèbre de diviseurs de K/k permet de définir $f(P)$ à partir d'un triple A, B, C tel que $f=f_{ABC}$. En effet, si X est le point de la droite (A, C) qui se divise par la plus grande puissance possible de P, on a $f(P)=a(A, B, C, X)$. Cette définition est valable dans toute algèbre de diviseurs, et l'auteur prouve que $f\to f(P)$ est un isomorphisme. Il en déduit que $\bar{k}$ est algébriquement fermé (démonstration: si $\varphi(x)$ est un polynome, et si $f_{DEF}=\varphi(f_{ABC})$, on a, pour tout $P|D$, $\varphi(f_{ABC}(P))=f_{DEF}(P)=0$). L'étude des conséquences des conditions (α) et (β) montre que, pour tout $f\epsilon\bar{K}$ non constant, tout $g\epsilon K$ est algébrique et de degré borné par rapport à $\bar{k}(f)$. Ainsi, pour toute algèbre de diviseurs, $\bar{k}$ est algébriquement fermé et $\bar{K}$ est un corps de fonctions algébriques d'une variable sur $\bar{k}$. On constate que cette algèbre de diviseurs coïncide avec celle de $\bar{K}/\bar{k}$. Ainsi, il y a une correspondance biunivoque entre les corps des fonctions algébriques d'une variable et les algèbres de diviseurs.

M. Krasner (Paris).

Nakayama, Tadasi. Note on 3-factor sets. Kōdai Math. Sem. Rep., no. 3, 11–14 (1949).

Let G be a finite group of automorphisms of a field K, F the fixed field of G, K^* the multiplicative group of K. Theorem 1. There is a homomorphism of the three-dimensional cohomology group of G in K^* into the two-dimensional cohomology group of G in the norm class group $F^*/N^*_{K/F}$. The homomorphism is given by the usual process of multiplying the coboundary relation over the group manifold; specifically if a is a three-cocycle, the corresponding two-cocycle is $\alpha(\mu, \nu)=b(\mu)b(\nu)b(\mu\nu)^{-1}\prod_\lambda a(\lambda, \mu, \nu)^{-1}$, where b is a one-cochain determined so that $b(\lambda)/\lambda b(\nu)=\prod_\mu a(\lambda, \mu, \nu)$. Such a b exists because the one-dimensional cohomology group of G in K^* vanishes; for this reason the theorem will not generalize to higher dimensions. Theorem 2. If H is a normal subgroup of G of order h, and L the corresponding subfield, the hth power of any three-dimensional cohomology class of G in K^* can be obtained by lifting up to G a suitable three-dimensional cohomology class of G/H in L^*. The reviewer conjectures that theorem 2 holds in all dimensions.

S. MacLane (Chicago, Ill.).

Nakayama, Tadasi. Halblineare Erweiterung des Satzes der Normalbasis und ihre Anwendung auf die Existenz der derivierten (differentialen) Basis. I. Proc. Japan Acad. 21 (1945), 141–145 (1949).

The author proves the following theorem. Let the field L be separable of finite degree over K, L^* the normal extension of L/K, and $\mathfrak{G}$ its Galois group. Let F be a subfield of L^* (not necessarily containing K, or of finite degree under L^*) and F^* the union field of the images of F under $\mathfrak{G}$. Then if (*) $(F^*K:F^*)\geqq(L^*K:K)$, the field L contains an element whose $(L:K)$ conjugates with respect to K are linearly independent over F. The proof uses the semilinear group ring $(\mathfrak{G}, F)$ [Nakayama and Shoda, Jap. J. Math. **12**, 109–122 (1936)] and involves much representation theory [Osima, Proc. Phys.-Math. Soc. Japan (3) **20**, 1–5 (1938); Nakayama, Amer. J. Math. **71**, 241–248 (1949); these Rev. **10**, 425]. This is used to generalize Riblet's theorem [Amer. J. Math. **63**, 347–351 (1941); these Rev. **2**, 346] on differential bases as follows. Without assuming characteristic zero, let L be separable of finite degree over K and possess an abstract differentiation, and let F be the field of constants. Then if (*) is satisfied, L has a differential basis over K. There is also a theorem on Galois-moduls of L^* with respect to $(\mathfrak{G}, F)$, generalizing results of Deuring [Math. Ann. **107**, 140–144 (1932)] and Nakayama [Ann. of Math. (2) **42**, 1–21 (1941); these Rev. **2**, 344].

G. Whaples (Bloomington, Ind.).

Azumaya, Gorô, and Nakayama, Tadasi. On absolutely uni-serial algebras. Jap. J. Math. 19, 263–273 (1948).

Let A be an algebra with unit element over K, N its radical, Z its center. Introductory sections are devoted to discussing decomposition into primary parts. The main theorem states that if A is primary, then every scalar extension of A is a direct sum of primary algebras if and only if the center of A/N is purely inseparable over $Z/(Z\cap N)$. We may call A uniserial if every right or left ideal is principal, absolutely uniserial if every scalar extension is uniserial. Two criteria for A to be absolutely uniserial are given, one being that (1) each primary component of Z can be generated by a single element, and (2) N is a principal ideal generated by an element of Z. In a final section it is shown that a primary algebra A over K is split and absolutely uniserial if and only if every representation of A over an extension field is equivalent to a representation over K.

I. Kaplansky (Chicago, Ill.).

Please see the note to p. 241. Tsuji on p. 871.

Azumaya, Gorô. On almost symmetric algebras. Jap. J. Math. 19, 329–343 (1948).

Symmetric, weakly symmetric, and Frobenius algebras are defined as by Nakayama [Ann. of Math. (2) **40**, 611–633 (1939); these Rev. **1**, 3]. The author first shows that A is a Frobenius algebra if and only if the right and left annihilators of the radical N are equal and can be generated by a single element c; if c can be chosen in the center, A is called almost symmetric. Almost symmetry lies in strength between symmetry and weak symmetry. It is shown in §§ 2, 3 that symmetry and almost symmetry are preserved under Kronecker products; moreover, A is strongly symmetric if and only if all its scalar extensions are weakly symmetric. These theorems clarify some results of Nakayama and Nesbitt [Ann. of Math. (2) **39**, 659–668 (1939)]. In § 4 a necessary and sufficient condition is given for a homomorphic image to be again almost symmetric; in § 5 results of Nakayama on semi-linear group algebras are extended. Some of the results are given for rings with chain condition as well as algebras, and in a final section an automorphism attached to a Frobenius ring is constructed.

I. Kaplansky (Chicago, Ill.).

Please see the note to p. 241, Tsuji on p. 871

Azumaya, Gorô. On generalized semi-primary rings and Krull-Remak-Schmidt's theorem. Jap. J. Math. 19, 525–547 (1948).

Chapter I is devoted to various remarks on the partially ordered set of idempotents in a ring. In chapter II a new definition is proposed for the radical of a ring A; if there exists a two-sided ideal C, containing all left and right ideals free of idempotents, then C is called the radical of A. Orthogonal idempotents in A/C can be lifted up to orthogonal idempotents in A. If the existence of minimal ideals in A/C is assumed, then certain stronger results can be proved. Chapter III. Let M be an Abelian group with operators Ω; suppose M is the direct sum of indecomposable Ω-subgroups M_i. The usual chain conditions are circumvented by the outright assumption that the sum of two non-automorphisms of each M_i is a non-automorphism; however, if the number of M_i's is infinite, it is assumed that each is finitely generated. Then it is shown that any two direct decompositions of M have isomorphic refinements. The ring of Ω-endomorphisms of M is given a detailed study in terms of the theory of chapter II. [Reviewer's remark. The radical proposed here coincides with the Jacobson radical under various weak chain conditions and also for compact rings. But it seems to be quite unsuited to applications such as Banach algebras. For example, the ring of continuous real functions on the unit interval has no Azumaya radical.]

I. Kaplansky (Chicago, Ill.).

Please see the note to p. 241, Tsuji on p. 871.

Thurston, H. A. Partly associative operations. J. London Math. Soc. 24, 260–271 (1949).

This paper is concerned with systems S in which a product $x_1\cdots x_{\nu+1}$ of "length" $\nu+1$ is defined. Such a system is termed weakly reversible if every element y of S is expressible as such a product, and regular if $x_1\cdots x_{i-1}ax_{i+1}\cdots x_{\nu+1}=x_1\cdots x_{i-1}bx_{i+1}\cdots x_{\nu+1}$ implies that $a=b$. An associative law in such a system is an identity of the type

$$x_1\cdots x_{i-1}(x_i\cdots x_{i+\nu})\cdots x_{2\nu+1}=x_1\cdots x_{i+j-1}(x_{i+j}\cdots x_{i+j+\nu})\cdots x_{2\nu+1}$$

$(1\leqq i<i+j\leqq\nu+1)$. This identity is abbreviated by the author to 'i'='$i+j$.' The main concern of the paper is to characterise the set of associative laws satisfied by a system S which is both weakly reversible and regular. The main result can be stated as follows. If S is both weakly reversible and regular, and further satisfies at least one associative law, then there exist integers p and q, p dividing q and q dividing ν, such that the associative law 'i'='$i+j$' is satisfied in S if and only if p divides i and q divides j. The system is then termed (p, q) associative. The author then shows by an example that (p, q) associative systems exist for all p, q satisfying the conditions indicated above.

D. Rees (Cambridge, England).

Albert, A. A. Almost alternative algebras. Portugaliae Math. 8, 23–36 (1949).

An almost left alternative algebra A over a field F of characteristic not two is defined by the identities

$$(1)\quad z(xy)=\alpha(zx)y+\beta(zy)x+\gamma(xz)y+\delta(yz)x+\epsilon y(zx)+\eta x(zy)+\sigma y(xz)+\tau x(yz)$$

for α, β, γ, δ, ϵ, η, σ, τ in F independent of x, y, z in A, (2) $xx^2=x^2x$, and the assumption that there exists an algebra B with unity quantity such that B satisfies (1) and B is not commutative. An almost alternative algebra is one which is both almost left alternative and almost right alternative (similarly defined). This paper classifies almost alternative algebras relative to quasiequivalence (B and A are quasiequivalent in F in case they are the same vector space and their products are related by $x\cdot y=\lambda xy+(1-\lambda)yx$ for $\lambda\neq\frac{1}{2}$ in F). The two principal residual types are the left alternative algebras [anti-isomorphic to the algebras in Ann. of Math. (2) **50**, 318–328 (1949); these Rev. **10**, 503] and new algebras which the author calls algebras of type (γ, δ). A partial structure theory of algebras of type (γ, δ) is given in the concluding section. *R. D. Schafer.*

Raffin, Raymond. Algèbres monosymétriques. C. R. Acad. Sci. Paris **230**, 31–33 (1950).

Certain elementary properties of finite-dimensional non-associative algebras over a field are extended to non-associative rings A which are vector spaces over a commutative (associative) ring B with unity element. A typical theorem is: if the principal (left and right) powers of each element of A coincide and if A has finite principal degree over B, then A is "monosymmetric" (that is, each element of A has a unique two-sided inverse [same C. R. **228**, 1685–1687 (1949); these Rev. **10**, 676]) if and only if (1) B is a field and (2) A does not contain elements $x\neq0$, $y\neq0$ such that $xy=yx=0$. *R. D. Schafer* (Philadelphia, Pa.).

Raffin, Raymond. Algèbres du troisième degré. C. R. Acad. Sci. Paris **230**, 164–166 (1950).

With a slight modification of terminology from the preceding note, the author considers the case where A is of finite dimension and third degree over B. *R. D. Schafer.*

Kaplansky, Irving. Topological representation of algebras. II. Trans. Amer. Math. Soc. **68**, 62–75 (1950).

L'auteur dit qu'un anneau A est π-régulier si, pour tout $a\epsilon A$, il existe un $x\epsilon A$ et un entier n (dépendant de a) tel que $a^nxa^n=a^n$. Il dit que A est fortement régulier si, pour tout a, il existe x tel que $a^2x=a$. Une algèbre algébrique (c'est-à-dire dont tout élément vérifie une équation algébrique à coefficient dans le corps de base) est un anneau π-régulier. L'auteur cherche d'abord à réduire l'étude des anneaux π-réguliers à celle des anneaux fortement réguliers; il étudie en particulier les anneaux π-réguliers qu'il appelle homogènes (nous renvoyons à l'article même pour la définition), et montre que lorsqu'un tel anneau admet un élément unité, c'est un anneau de matrices sur un anneau fortement régulier. Il examine ensuite la structure des algèbres algébriques et fortement régulières; si en outre, pour une telle algèbre A, et pour tout idéal maximal M, A/M est d'ordre fixe n^2 sur son centre, alors A est somme directe d'un nombre fini d'algèbres, dont chacune est produit kroneckérien (sur une extension du corps de base) d'une algèbre commutative et d'un corps non commutatif de rang fini. Ces résultats permettent à l'auteur d'obtenir un résultat nouveau dans le problème de Kurosch en montrant que toute algèbre algébrique satisfaisant une identité polynomiale est localement finie. *J. Dieudonné* (Nancy).

Iwasawa, Kenkichi. On the representation of Lie algebras. Jap. J. Math. **19**, 405–426 (1948).

Let L be a Lie algebra, and let N be an Abelian ideal in L. Then N is said to split L if L is the direct sum of N and of a subalgebra L_1 of L. The author first proves that it is always possible to find a Lie algebra L_0 and an Abelian ideal N_0 in L_0 with the following properties: L_0 contains L, $N_0\cap L=N$, $L_0=N_0+L$, N_0 splits L_0; L_0 is then called a splitting algebra of L (relatively to N). To begin with, a splitting algebra is constructed in the most general case which, however, is always infinite dimensional (except if $N=L$). The author then proceeds to extract (in the case where L is finite dimensional) a finite dimensional splitting algebra from the infinite dimensional one. In order to do this, he treats separately the cases where the basic field is of characteristic 0 or not 0. In the first case, he considers successively the cases where L is nilpotent, solvable or general; for the first two cases, he makes extensive use of a process analogous to the "straightening process" of G. Birkhoff [Ann. of Math. (2) **38**, 526–532 (1937)]. In the case of characteristic p, L is first imbedded in a restricted Lie algebra in the sense of Jacobson [Trans. Amer. Math. Soc. **42**, 206–224 (1937)]. Once the existence of a finite dimensional splitting for a (finite dimensional) Lie algebra L with respect to its center is established, it is not difficult to derive a proof of Ado's theorem, to the effect that L admits a faithful linear representation. *C. Chevalley.*

Please see the note to p. 241, Tsuji on p. 871

Hochschild, G. Lie algebras and differentiations in rings of power series. Amer. J. Math. **72**, 58–80 (1950).

E. Cartan has given a transcendental proof of Ado's theorem that every Lie algebra over the field of complex numbers has some faithful representation [J. Math. Pures Appl. (9) **17**, 1–12 (1938)]. The object of the present paper is to give an algebraic proof of the same theorem (except that the basic field is not assumed to be the field of complex numbers, but only to be of characteristic 0) following the main idea of Cartan's transcendental proof. Let L be a Lie algebra over a field of characteristic 0; then the exterior algebra G over the dual space of L (the algebra of cochains of G in the terminology of cohomology theory) has an antiderivation d of square 0. The first main result is that G can be mapped isomorphically into the algebra of exterior differential forms on a ring of power series $K\langle x_1, \cdots, x_n\rangle$ in a certain number of independent variables $x_1, \cdots, x_n$ in such a way that the antiderivation d corresponds to the operation of exterior differentiation of differential forms. Such an isomorphism defines by duality a representation of L by derivations of $K\langle x_1, \cdots, x_n\rangle$. It is further proved that the isomorphism may be constructed in such a way that the derivations which represent the elements of L map some finite dimensional subspace M of $K\langle x_1, \cdots, x_n\rangle$ into itself and that the representation thus defined of L by endomorphisms of M is faithful. The proof is first carried through in the case where L is solvable (it is not necessary to consider separately the case where L is nilpotent). Then, having represented a solvable L as an algebra of derivations of $K\langle x_1, \cdots, x_n\rangle$, one shows that, to any given derivation D of L one may associate a derivation D' of $K\langle x_1, \cdots, x_n\rangle$ such that $[D', L]\subset L$ and that the effect of bracketing with D' in L is identical with operating by D. The operation D' is not uniquely determined; but, if S is a semi-simple algebra of derivations of L, it is nevertheless possible to associate a D' to every D in S in such a way that the operations D' form an algebra isomorphic with S and map some finite dimensional subspace of $K\langle x_1, \cdots, x_n\rangle$ into itself. *C. Chevalley* (New York, N. Y.).

Wessel, Walter. On infinite relativistic particle matrices. Physical Rev. (2) **76**, 1512–1519 (1949).

This paper is concerned with obtaining infinite-dimensional unitary representations of a Lie algebra consisting of 16 linearly independent elements. Six of these are the generators of the Lorentz group. Two are scalars formed from these six quantities. The remaining eight are components of two four-dimensional vectors which the author interprets as the four-velocity and spin vector and proposes to base a theory of particles on this interpretation. It is shown that these sixteen quantities can be built up from bilinear expressions formed from two two-component spinors. The representations are then determined from the representations of the spinors. *A. H. Taub* (Urbana, Ill.).

THEORY OF GROUPS

Šubnikov, A. V. On the symmetry of vectors and tensors. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **13**, 347–375 (1949). (Russian)

It is useful in crystallography to distinguish between the symmetry of geometrical figures and of "material" figures, which are figures with certain additional properties of a physical nature. There is, for instance, a difference in the symmetry of a cube with six white faces and a cube with five white faces and a black one. The crystal cubes of rock salt have the optical symmetry of ordinary spheres, but those of sodium chlorate that of spheres without symmetry planes but with a left or right orientation according to the rotation character of the plane of polarization. Since many physical phenomena in crystallography can be described with the aid of vectors and tensors the author proposes in this paper to establish the concept of the symmetry of vectors and tensors. Starting with the matrices c_{ij}, $i, j=1, 2, 3$, which describe the different rotations and reflections of a three-dimensional system of orthogonal Cartesian axes he systematically tests the symmetry properties of polar vectors, bivalent polar tensors, of axial vectors (bivalent polar tensors), and of bivalent axial tensors. The symbols to indicate the groups of symmetry are ∞ for an axis of infinite order, t for a plane of symmetry and * for parallelism, etc. The notation will be clear from the following table indicating the symmetries of polar tensors.

Symmetry of the tensor	Form of the tensor	Situation of the axes
$\bar{2}$	$\begin{cases} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{cases}$	arbitrary
$2:t$	$\begin{cases} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{cases}$	axis 2 coincides with the x_3-axis
$t\cdot 2:t$	$\begin{cases} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{cases}$	axes 2 coincide with the x_1, x_2, x_3 axes
$\infty:t$	$\begin{cases} a_{11} & -a_{21} & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & a_{33} \end{cases}$	axis ∞ coincides with the x_3-axis
$t\cdot\infty:t$	$\begin{cases} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{33} \end{cases}$	axis ∞ coincides with the x_3-axis
$\infty/\infty:t$	$\begin{cases} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{cases}$	arbitrary

There are similar tables for the other tensors investigated. For further literature the author refers, e.g., to his book "Symmetry" [Moscow-Leningrad, 1940] and a paper unavailable to the reviewer, as well as to the papers by F. Seitz [Z. Kristallogr., Mineral. Petrogr. Abt. A. **88**, 433–459 (1934); **90**, 289–313 (1935); **91**, 336–366 (1935); **94**, 100–130 (1936)]. *D. J. Struik* (Cambridge, Mass.).

Šubnikov, A. V. Symmetry and geometrical interpretation of two-dimensional polar tensors. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **13**, 376–391 (1949). (Russian)

A bivalent symmetrical tensor in two dimensions a_{ik} is geometrically interpreted by the conic section with the equation $a_{11}x_1^2+a_{22}x_2^2+2a_{12}x_1x_2=1$. The author proposes a method for dealing in a similar geometrical way with non-symmetrical tensors. For this purpose he connects two vectors p_i, q_i by the relation $p_i=a_{ij}q_j$, $i, j=1, 2$ and considers the circle $q_1^2+q_2^2=1$ together with its transform

$$p_1^2(a_{22}^2+a_{21}^2)+p_2^2(a_{11}^2+a_{12}^2)-2p_1p_2(a_{22}a_{12}+a_{11}a_{21}) =(a_{11}a_{22}-a_{21}a_{12})^2,$$

connecting corresponding points on the two curves by straight lines. The resulting figures show the existing symmetries. For instance, the tensor given by $a_{11}=1.5$, $a_{12}=.2$, $a_{21}=.3$, $a_{22}=2.0$ has the symmetry 2 [see the preceding review], the tensor $a_{11}=1.61$, $a_{12}=a_{21}=.21$, $a_{22}=1.85$ has the symmetry $2\cdot t$, the tensor $a_{11}=a_{22}=1.5$, $a_{12}=a_{21}=0$ has the symmetry $\infty\cdot t$. The following table gives a summary:

Symmetry of the tensor	Form of the tensor	Coordinate system
2	$\begin{cases} a_{11} & a_{12} \\ a_{21} & a_{22} \end{cases}$	arbitrary
$2\cdot t$	$\begin{cases} a_{11} & 0 \\ 0 & a_{22} \end{cases}$	principal axes
∞	$\begin{cases} a_{11} & a_{12} \\ -a_{12} & a_{11} \end{cases}$	arbitrary
$\infty\cdot t$	$\begin{cases} a_{11} & 0 \\ 0 & a_{11} \end{cases}$	arbitrary

D. J. Struik (Cambridge, Mass.).

Rédei, L. Kurzer Beweis des gruppentheoretischen Satzes von Hajós. Comment. Math. Helv. **23**, 272–282 (1949).

Let G be a finite Abelian group with unit element 1. A set of elements of G: $1, \alpha, \cdots, \alpha^{e-1}$, where e is greater than unity and less than the order of α, is called a simplex and is denoted by $[\alpha]_e$; if e is prime then $[\alpha]_e$ is called a prime simplex. If H and K are two complexes consisting of m and n elements of G, respectively, the product HK is defined to be the direct product of H and K when this product contains mn different elements. Hajós has proved the following theorem. (H) If G can be represented as a

product

$$(1)\qquad G=[\alpha_1]_{e_1}[\alpha_2]_{e_2}\cdots[\alpha_n]_{e_n},\qquad n\geqq 1,$$

then at least one simplex $[\alpha_i]_{e_i}$ is a subgroup of G [Math. Z. 47, 427–467 (1941); these Rev. 3, 302; see also L. Rédei, Acta Univ. Szeged. Sect. Sci. Math. 13, 21–35 (1949); these Rev. 10, 683]. This theorem provides the proof of the Minkowski conjecture concerning linear forms and can be stated in several equivalent forms other than the above group theoretical one [see the above references]. The present paper contains a proof of (H) which is simpler than that given by Hajós. The main steps in the proof are as follows. (i) If (H) is true when each factor in (1) is a prime simplex it is true in general. (ii) The ("ordinary") complexes of G may be regarded as forming a subset of the group ring Γ of G over the ring of the rational integers; the members of Γ are called "general" complexes. Thus the simplex $[\alpha]_e$ corresponds to the member $\gamma=1+\alpha+\cdots+\alpha^{e-1}$ of Γ. Let $\bar{\alpha}$ denote either γ, for a prime e, or $1-\alpha$; let $\{K_1,\cdots,K_r\}$ denote the group generated by the members of the general complexes $K_1,\cdots,K_r$ and let $g\{K_1,\cdots,K_r\}$ be the number of prime factors, not necessarily different, dividing the order of this group. Then if the equation $K\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m=0$ holds in Γ where K is a general complex, and if no factor $\bar{\alpha}_i$ may be omitted without affecting the truth of this equation, it is shown that

$$g\{K,\alpha_1,\alpha_2,\cdots,\alpha_m\}-g\{K\}<m.$$

(iii) If the product $K[\alpha_1]_{p_1}\cdots[\alpha_r]_{p_r}$ is a group, where $p_1,\cdots,p_r$ are prime and K is an ordinary complex and not a group, then $g\{K,\alpha_1,\cdots,\alpha_r\}-g\{K\}<r$. (iv) Proof of (H) from (i), (ii) and (iii). The most difficult part of the proof is the demonstration of (ii) which is effected by induction on $g\{\alpha_1\}+g\{\alpha_2\}+\cdots+g\{\alpha_m\}$. *R. A. Rankin.*

Rédei, L. Die Reduktion des gruppentheoretischen Satzes von Hajós auf den Fall von p-Gruppen. Monatsh. Math. 53, 221–226 (1949).

It is shown that it is sufficient to prove the theorem (H) of the preceding review for the case when the group G is a prime power group. If the general group G can be represented as in (1) then it can also be represented as the product

$$(2)\qquad G=[\gamma_1\beta_1]_{p_1}\cdots[\gamma_m\beta_m]_{p_m}[\rho_1]_{r_1}\cdots[\rho_v]_{r_v},$$

where γ_i, β_i are of orders p_i and q_i ($p_i\neq q_i$; $i=1,2,\cdots,m$) and ρ_j is of order $r_j^{c_j}$ ($c_j\geqq 2$; $j=1,2,\cdots,v$); here p_i, q_i and r_j are prime. In this representation the elements β_i and ρ_j may be replaced by $\beta_i^{k_i}$ and $\rho_j^{l_j}$, where k_i is arbitrary and l_j is prime to r_j. The assumption is made that (H) is false for G but true for prime power groups. By choosing k_i and l_j suitably it is shown that the unit element can be expressed as a product of different elements $\gamma_i\beta_i$ and ρ_j, and therefore occurs twice on the right of (2), which gives a contradiction. The proof, which is not easy, is inductive and makes no use of the group ring Γ. *R. A. Rankin.*

Szele, T. Neuer vereinfachter Beweis des gruppentheoretischen Satzes von Hajós. Publ. Math. Debrecen 1, 56–62 (1949).

A proof is given of Hajós's theorem (H) [see the second preceding review]. The author succeeds in dispensing with Rédei's result (iii) by showing that, when the factors in (1) are prime simplexes, $g\{\alpha_1,\alpha_2,\cdots,\alpha_k\}\geqq k$ for each $k\leqq n$. This simplification enables him to rearrange the ordering of the proof so that the part (ii), which is given as in Rédei's first paper above, comes last and is the only part in which the group ring Γ is needed. The conjecture is made that (H) holds for all Abelian torsion groups. *R. A. Rankin.*

Fáry, István. Die Äquivalente des Minkowski-Hajósschen Satzes in der Theorie der topologischen Gruppen. Comment. Math. Helv. 23, 283–287 (1949).

Let G be an additive commutative topological group. Let $\alpha(x)$ be a variable element of a one parameter subgroup of G. Then the set of elements $\alpha(x)$ for $\xi\leqq x<\eta$, where $-\infty<\xi\leqq 0<\eta<+\infty$, is called a group interval and is denoted by $[\alpha]$. Here ξ and η must be such that $\alpha(x)\neq\alpha(y)$ for $\xi\leqq x\neq y<\eta$. The set $[\alpha]$ is a group if and only if $\alpha(\xi)=\alpha(\eta)$. The author proves the following analogue of (H) [see the third preceding review] for topological groups. (F) If G can be expressed as a direct sum of group intervals $G=[\alpha_1]+\cdots+[\alpha_n]$, i.e., if every $\beta\epsilon G$ is uniquely expressible in the form

$$(3)\qquad \beta=\alpha_1(x_1)+\cdots+\alpha_n(x_n),\qquad \xi\leqq x_i<\eta;\ i=1,2,\cdots,n,$$

then at least one group interval is a group. Just as (H) is a kind of converse of the fundamental theorem for finite Abelian groups so is (F) a kind of converse of the corresponding theorem for compact locally connected and connected commutative groups. In order to deduce (F) the author goes back to one of the equivalent forms of (H), namely Hajós's "zweite Fassung" [this is theorem (Ic) quoted in the review of the paper by Rédei cited in the 3d preceding review] which he shows is equivalent to (F). By means of (3) each member of the group can be made to correspond to that point in n-dimensional space whose coordinates are $(x_1,x_2,\cdots,x_n)$ referred to the basis vectors of the lattice. These points occupy the parallelepiped $\xi\leqq x_i<\eta$ $(i=1,2,\cdots,n)$ and it is to the lattice of such parallelepipeds that Hajós's theorem (in the form (Ib)) is applied. *R. A. Rankin* (Cambridge, England).

Piccard, Sophie. Classes de substitutions d'un groupe imprimitif. C. R. Acad. Sci. Paris 229, 693–695 (1949).

Piccard, Sophie. Les classes de substitutions des groupes imprimitifs. II. C. R. Acad. Sci. Paris 229, 739–741 (1949).

Piccard, Sophie. Les classes de substitutions d'un groupe imprimitif et les bases d'un groupe imprimitif saturé. C. R. Acad. Sci. Paris 229, 1193–1195 (1949).

Consider an imprimitive permutation group G of degree $n=kp$, where the number of sets of imprimitivity E_i into which the n symbols can be divided is p. Denote by H the self-conjugate subgroup of G whose operations leave each of the p sets E_i unchanged. Then $G/H\sim g$, and with each permutation S of G on the symbols $1,2,\cdots,n$ is associated a permutation σ of g on the symbols E_i $(i=1,2,\cdots,p)$. If S has α_i cycles of order i then $n=\alpha_1+2\alpha_2+\cdots$ and the number of cycles of S is given by $q=\alpha_1+\alpha_2+\cdots$; denote the number of cycles of σ by s. The permutations S of G are distinguished as even or odd according to the parity of $n\pm q$ in the familiar theory. The author establishes three sets of criteria generalizing this division: (a) a permutation S is said to belong to class C_1 if the quantity $(k+1)s+p+q$ is even and to C_2 if this quantity is odd; (b) a permutation S is said to belong to class C_3 if $ks+q$ is even and to C_4 if $ks+q$ is odd; (c) finally, a classification of the elements S can be made according as (i) S is even or odd, (ii) S and σ have the same or different parity, and (iii) σ is even or odd, all in the familiar sense referred to above. These various classifications may overlap, but resemble the familiar classification in their general properties. *G. de B. Robinson.*

Piccard, Sophie. Sur les bases du groupe alterné. Comment. Math. Helv. **23**, 123–152 (1949).

The alternating group A_n ($n \geqq 4$) can be generated by two substitutions S, T called a basis; such a basis can of course be chosen in a great variety of ways. After proving several preliminary theorems which are applicable if $R^{-1}SR = T$, where R is an even or odd substitution of order two, the author lists all possibilities for $n = 4, 5, 6, 7$.

G. de B. Robinson (Toronto, Ont.).

Garnir, H. Sur la détermination des caractères primitifs d'un groupe d'ordre fini. Bull. Soc. Roy. Sci. Liége **18**, 190–202 (1949).

In this note the author discusses the determination of the irreducible characters (over the field of complex numbers) of a finite group, if one supposes as known in advance the multiplication table of the classes of conjugates of the group. Neither the results obtained nor the methods used are new; the presentation is self-contained and elementary.

R. M. Thrall (Ann Arbor, Mich.).

Sanov, I. N. Application of Lie rings to the theory of periodic p-groups. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 180 (1949). (Russian)

In this abstract, the author announces further results (and also corrections) concerning work of Magnus and of Grün. Local finiteness of Lie rings is connected with a restricted form of Burnside's problem. The Lie rings generated by $[uv^n] = 0$, $n = 2, 3$ (and $n = 4$, with characteristic not 5), are announced to be locally finite. Specific references are not given in the abstract. See, however, W. Magnus [J. Reine Angew. Math. **182**, 142–149 (1940); these Rev. **2**, 214] and O. Grün [ibid., 158–177 (1940); these Rev. **2**, 212].

F. Haimo (St. Louis, Mo.).

Frucht, Robert. On the construction of partially ordered systems with a given group of automorphisms. Amer. J. Math. **72**, 195–199 (1950).

Continuing work of G. Birkhoff [Revista Unión Mat. Argentina 11, 155–157 (1946); these Rev. **7**, 411] and the author [ibid. **13**, 12–18 (1948); these Rev. **9**, 409], it is shown that if G is a finite group of order g generated by n elements, then there exists a partially ordered set P with $(n+2)g$ elements whose group of automorphisms is isomorphic to G. If $n > 2$ then $(n+2)g$ can be replaced by $(m+1)g$ where m is the least integer with $m \geqq \frac{1}{2}[1+(8n+1)^{\frac{1}{2}}]$. Such P can also be found with at most $(2+\log g/\log 2)g$ elements.

P. M. Whitman (Silver Spring, Md.).

Tits, J. Généralisation des groupes projectifs. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 568–589 (1949).

In two previous articles bearing the same title [same vol., 197–208, 224–233 (1949); these Rev. **11**, 9] the author studied triply transitive groups (in particular, projective and semi-projective groups) without making any assumptions concerning the space in which the elements of the group operate. Concerned now with the case where the space consists of a finite number $N+1$ of points, he constructs all (finite) triply transitive groups. If $N = 2^n$ or p^{2k+1}, they number $(N-2)!/n$ or $(N-2)!/(2k+1)$, respectively (all projective); if $N = p^{2k}$ ($p \neq 2$) there are $(N-2)!/2k$ projective and $(N-2)!/2k$ nonprojective. For other values of N none exist.

L. M. Blumenthal (Columbia, Mo.).

Tits, J. Généralisations des groupes projectifs. IV. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 756–773 (1949).

If E is a set of $N+1$ points over which a triply transitive group of transformations is defined, then $N = p^n$ (p, a prime). The author has shown in his third communication [reviewed above] that (a) when $p = 2$ or $n = 2k+1$, all the triply transitive groups operating in E are projective, while (b) for $p > 2$ and $n = 2k$ both projective and nonprojective such groups exist in E. Let E_H, E_G denote, respectively, the spaces obtained when a projective group H or a triply transitive nonprojective group G operates in E. In (b) a one-to-one correspondence can be established between H and G, as also between associated spaces E_H and E_G. The author studies properties of the groups H, G and the associated spaces.

L. M. Blumenthal (Columbia, Mo.).

***Ferguson, William Allen. On the Classification of Finite Metabelian Groups with Six Generators.** Abstract of a Thesis, University of Illinois, 1946. ii+13 pp.

The author continues the work of H. R. Brahana [cf. Amer. J. Math. **62**, 365–379 (1940); these Rev. **1**, 257] on metabelian groups G in which (1) the commutator subgroup is the center and (2) $S^p = 1$ for all $S \epsilon G$, p a prime. In the present paper the groups G with 6 generators and orders p^{15}, p^{14}, or p^{13} are determined by geometric methods; the classification is made by a study of the relative positions of commutators and noncommutators in the commutator subgroup.

R. M. Thrall (Ann Arbor, Mich.).

***Seybold, Mary Anice. Isomorphism Groups of Metabelian Groups Generated by Four Independent Operators of Order p.** Abstract of a Thesis, University of Illinois, 1947. 7 pp.

The author continues the work of H. R. Brahana [Amer. J. Math. **62**, 365–379 (1940); these Rev. **1**, 257] on metabelian groups G in which (1) the commutator subgroup is the center, (2) $S^p = 1$ for all $s \epsilon G$, p a prime, (3) G has four generators. Brahana has shown that there are twelve such groups, all of which are factor groups of one, called the master group; the author determines the group of automorphisms for each of these groups. The method of attack is geometric.

R. M. Thrall (Ann Arbor, Mich.).

Szekeres, G. Determination of a certain family of finite metabelian groups. Trans. Amer. Math. Soc. **66**, 1–43 (1949).

The author has recently [Ann. of Math. (2) **49**, 43–52 (1948); these Rev. **9**, 492] classified by means of numerical invariants all groups which are extensions of an elementary Abelian group by a cyclic group. In the present paper he applies the same method to obtain a complete determination of all groups $\mathfrak{G}$ which are extensions of an arbitrary Abelian group $\mathfrak{A}$ of order h by a cyclic group $\mathfrak{C} \cong \mathfrak{G}/\mathfrak{A}$ of order n, with the sole restriction that the greatest common divisor (h, n) shall be square-free. The method consists again in considering $\mathfrak{C}$ as an operator-domain of which a ring of endomorphisms of $\mathfrak{A}$ is supposed to be a homomorphic image. Let $\mathfrak{J}[x]$ be the ring of polynomials in a variable x with integer coefficients and $\mathfrak{R}_{n,h}[x]$ its finite residue-class ring $\mathfrak{J}[x]/(x^n-1, h)$. The ring of endomorphisms of $\mathfrak{A}$ should then contain a subring $\mathfrak{R}$ which is a homomorphic image of $\mathfrak{R}_{n,h}[x]$. Rather than starting from a given Abelian group $\mathfrak{A}$ and finding all suitable rings $\mathfrak{R}$ the author now reverses the problem: he sets out to characterize for a given homomorphic image $\mathfrak{R}$ of $\mathfrak{R}_{n,h}[x]$ all those finite

Abelian groups $\mathfrak{A}$ which admit $\mathfrak{R}$ as a ring of endomorphisms, in other words all finite $\mathfrak{R}$-modules $\mathfrak{A}$.

In section 1 the structure of indecomposable constituents of $\mathfrak{A}$ is investigated. By the Krull-Schmidt theorem their types and multiplicities determine, and are determined by, $\mathfrak{A}$. A first decomposition splits $\mathfrak{A}$ into its p-primary components $\mathfrak{A}^{(p)}$ with operator-ring $\mathfrak{R}_p$, a homomorphic image of $\mathfrak{R}_{n,p}f[x]$. The transition from $\mathfrak{J}[x]$ to the ring of polynomials with p-adic integer coefficients $\mathfrak{P}[x]$ and, correspondingly, from $\mathfrak{R}_{n,p}f[x]$ and $\mathfrak{R}_p$ to their p-adic equivalents formally facilitates the subsequent work. Let now $n=p^k\cdot m$, $(m,p)=1$. A decomposition of x^m-1 into irreducible factors over $\mathfrak{P}[x]$, say $x^m-1=\varphi_0\cdot\varphi_1\cdot\cdots\cdot\varphi_t$, $\varphi_0=x-1$ leads to a factorisation $x^n-1=\Phi_0\cdot\Phi_1\cdot\cdots\cdot\Phi_t$, where $\Phi_\nu\equiv\varphi_\nu^{p^k}\pmod p$ and to a direct decomposition of $\mathfrak{A}^{(p)}$: $\mathfrak{A}^{(p)}=\mathfrak{A}_0^{(p)}\oplus\mathfrak{A}_1^{(p)}\oplus\cdots\oplus\mathfrak{A}_t^{(p)}$. Here $\mathfrak{A}_\nu^{(p)}$ is annihilated by Φ_ν and, as theorem I shows, also by a suitable power i of φ_ν. If now $k=0$, i.e., $(n,p)=1$, then $i=1$ [there is a misprint on p. 7, line 1] and the situation is similar to that encountered in the previous paper [loc. cit.]. An indecomposable $\mathfrak{A}_\varphi$ belonging to an irreducible φ of degree d is generated by a single element of order p^l, and as an abstract group $\mathfrak{A}_\varphi$ has type $(p^l,p^l,\cdots,p^l)$ and order p^{ld}. But if n is divisible by p serious difficulties arise which the author has succeeded in overcoming only for $k=1$. Even here the proof of the main theorem which elucidates the structure of the indecomposable $\mathfrak{R}_\varphi$-modules $\mathfrak{A}_\varphi$ (which are no longer generated by a single element) occupies more than twenty pages [sections 4 and 5] and cannot be described briefly. In section 2 a canonical form is developed for the element of $\mathfrak{A}$ which plays the role of the "factor-system" in Schreier's extension theory. Also possible isomorphisms between the groups $\mathfrak{G}$ so constructed and the avoidance of duplications are fully treated. Section 3 deals with applications. Among these we mention the construction of all soluble groups in which the normaliser of every element other than 1 is Abelian [see L. Weisner, Bull. Amer. Math. Soc. **31**, 413–416 (1925)]; a survey of all non-Abelian groups whose proper subgroups are all Abelian [see L. Rédei, Comment. Math. Helv. **20**, 225–264 (1947); these Rev. **9**, 131]; the construction of all p-groups with Abelian subgroups of index p. This latter problem, of which many special cases are known, is treated in some detail and serves as a good illustration of the foregoing methods and results. Asymptotic estimates for the number of such groups for fixed n and large p and for fixed p and large n are given.

K. A. Hirsch (Newcastle-upon-Tyne).

Szele, Tibor. Sur la décomposition directe des groupes abéliens. C. R. Acad. Sci. Paris **229**, 1052–1053 (1949).

The author gives a simple proof of the following theorem. If the Abelian group A contains an element, not 0, of finite order, then A is directly indecomposable if, and only if, A is either cyclic of order a power of a prime or else A is of type p^∞ [in the sense of Prüfer]. Only special cases of this theorem had been known before.

R. Baer.

Baer, Reinhold, and Williams, Christine. Splitting criteria and extension types. Bull. Amer. Math. Soc. **55**, 729–743 (1949).

The first-named author has recently [Amer. J. Math. **71**, 461–490 (1949); these Rev. **10**, 506] studied the partially ordered set of extension types of an Abelian group and has given (under suitable restrictions imposed on the extensions) an "internal" characterisation of this set by means of Loewy-chains of the Abelian group. This work is carried over in the present paper to the more general case of admissible extensions of (non-Abelian) operator-groups. The generalisation becomes possible by imposing restrictions (a) on the homomorphisms considered: they have to carry normal admissible subgroups into normal admissible subgroups ("normality preserving" homomorphisms); (b) on the extensions considered: they correspond to the restrictions on "little" extensions in the Abelian case. The ideas, concepts, methods, and results are easily recognized as straightforward generalisations of those developed for Abelian groups. But first the authors have to adapt and extend the theory of splitting endomorphisms [Baer, Trans. Amer. Math. Soc. **61**, 508–516 (1947); these Rev. **8**, 563] to the present case. In an appendix the authors deal with a special case of Hopf's problem on the existence of groups with finite numbers of generators which are isomorphic to proper quotient groups. It is shown that a finitely generated operator-group G for which every normality preserving admissible endomorphism of G splits G cannot be isomorphic to a proper quotient-group.

K. A. Hirsch.

Čunihin, S. A. On the conditions of theorems of Sylow's type. Doklady Akad. Nauk SSSR (N.S.) **69**, 735–737 (1949). (Russian)

The author pursues his extensions of theorems of Sylow, Hall, and Schur and Zassenhaus concerning composition and normal series, weakening the restriction on the invariance of the subgroups. For notation, see the review of his earlier paper [same Doklady (N.S.) **66**, 165–168 (1949); these Rev. **10**, 678], with Π fixed. Let series (1) be $\mathfrak{G}=\mathfrak{G}_0\supset\mathfrak{G}_1\supset\cdots\supset\mathfrak{G}_s=\mathfrak{E}$, where the subgroup $\mathfrak{G}_i$ has order g_i. For fixed i, the totality of all subgroups $\mathfrak{A}$ of the group $\mathfrak{G}_{i-1}$ which contain $\mathfrak{G}_i$ is called the ith factor-structure of (1) and denoted by $[\mathfrak{G}_{i-1},\mathfrak{G}_i]$. Let m_i be the divisor of g_{i-1}/g_i, maximal as described in the cited review. If $[\mathfrak{G}_{i-1},\mathfrak{G}_i]$ contains at least one subgroup $\mathfrak{M}_i$ of order m_ig_i having a normal series from $\mathfrak{M}_i$ to $\mathfrak{G}_i$ with every index a prime in Π, then $[\mathfrak{G}_{i-1},\mathfrak{G}_i]$ is said to be of type Π. If all subgroups of order m_ig_i contained in $[\mathfrak{G}_{i-1},\mathfrak{G}_i]$ of type Π are conjugate in $\mathfrak{G}_{i-1}$, then $[\mathfrak{G}_{i-1},\mathfrak{G}_i]$ is said to be of type $\Pi-1$. A subgroup $\mathfrak{H}$ is called Π-permutable in the finite group $\mathfrak{G}$ provided either $\mathfrak{H}$ permutes with every Sylow subgroup of $\mathfrak{G}$ of order dividing m or else $m=1$. A series (1) is a Π-permutable series of $\mathfrak{G}$ provided every $\mathfrak{G}_i$ is Π-permutable in $\mathfrak{G}_{i-1}$. A necessary and sufficient condition for $[\mathfrak{G},\mathfrak{E}]$ to be of type Π is that $\mathfrak{G}$ contains a subgroup $\mathfrak{G}_1$ such that $[\mathfrak{G},\mathfrak{G}_1]$ is of type Π and $[\mathfrak{G}_1,\mathfrak{E}]$ is of type $\Pi-1$; $[\mathfrak{G},\mathfrak{E}]$ is of type $\Pi-1$ if and only if $\mathfrak{G}$ contains a Π-permutable series, of which every factor-structure is of type $\Pi-1$.

R. A. Good (College Park, Md.).

Myagkova, N. N. On groups of finite rank. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 495–512 (1949). (Russian)

A group has general rank R if every finite subset is contained in a subgroup with R generators; it has special rank r if every finite subset generates a group with r generators. For Abelian groups, $r=R$, and Abelian groups of finite rank have been classified. In this paper the author investigates non-Abelian groups of finite rank that satisfy various additional assumptions. In § 1 some preliminary remarks are collected. In § 2 results are obtained concerning solvable groups of finite special rank (solvability meaning the existence of a well ordered composition series with Abelian factor groups). An example shows that a solvable group of

finite general rank may have infinite special rank. On the other hand, for nilpotent groups one has the explicit estimate $r \leq R+R^2+\cdots+R^k$, where k is the length of the descending central series. In § 3 it is shown that a locally finite p-group of finite special rank is countable, solvable, and has the descending chain condition on subgroups. In § 4 locally nilpotent groups of finite rank are considered. If such a group has no elements of finite order it is actually nilpotent; and in general it is swept out by its ascending central series. The proofs in this final section lean heavily on results of Mal'cev [same vol., 201–212 (1949); these Rev. 10, 507]. *I. Kaplansky* (Chicago, Ill.).

Zappa, Guido. Sulla condizione perchè un omomorfismo ordinario sia anche un omomorfismo strutturale. Giorn. Mat. Battaglini (4) 2(78), 182–192 (1949).

A join-homomorphism of group G onto group G' is a single-valued correspondence carrying each subgroup H of G into a subgroup H' of G' such that $(H\cup J)'=H'\cup J'$. Any element-homomorphism induces a join-homomorphism; any join-homomorphism so obtained is called normal. A meet-homomorphism is defined dually, and is called normal if induced by mapping $H\subset G$ into the subgroup of G' isomorphic to $\bar{G}\cap H$, where $\bar{G}$ is a fixed subgroup of G. A join-homomorphism which is also a meet-homomorphism is a lattice-homomorphism. Theorem: for there to exist a correspondence between finite groups G and G' which is both a normal join- and normal meet-homomorphism it is necessary and sufficient that $G=N\times H$, where N and H are of relatively prime orders and H is isomorphic to G' (this lattice-homomorphism maps $S\subset G$ into the isomorph of $S\cap H$). Given a finite group G and a normal subgroup N, the normal join-homomorphism between G and G/N is also a meet-homomorphism if and only if (letting h be the greatest divisor of the order of N which is relatively prime to the index of N), there exist subgroups $H\subset N$ of order h and $T\subset G$ such that $G=T\times H$, $R=T\cap N$ is cyclic and contained in the center of T, and every Sylow subgroup of T having $T\cap R>1$ is cyclic or is generated by g and k with $g^{2^{r-1}}=1$, $k^2=g^{2^{r-2}}$, $k^{-1}gk=g^{-1}$. Some special cases are considered. *P. M. Whitman* (Silver Spring, Md.).

Reidemeister, Kurt. Über Identitäten von Relationen. Abh. Math. Sem. Univ. Hamburg 16, 114–118 (1949).

The following general theorem is proved. Let R be a multiplicative group and S a multiplicative group of operators on R (operator symbol: $\cdot$). Let $\varphi: R\rightarrow S$ be a homomorphism with $\varphi(s-r)=s^{-1}\varphi(r)s$. Let k be the kernel of φ, and P the smallest normal subgroup in R containing all elements of form $r_1^{-1}r_2r_1[\varphi(r_1)\cdot r_2]^{-1}$. Finally let $\mathfrak{A}$ be the smallest normal subgroup in R containing the commutator subgroup and P. Then, if there exists a subgroup $H\subset R$ mapped isomorphically on $\varphi(R)$ by φ [reviewer's remark: this condition is equivalent to requiring k to be a direct summand of R], one has $\mathfrak{A}\cap k=P$. This result is applied to groups given by generators and defining relations, and a result on identities of relations, using "homotopy chains," is obtained. *J. Dugundji* (Los Angeles, Calif.).

Peiffer, Renée. Über Identitäten zwischen Relationen. Math. Ann. 121, 67–99 (1949).

Let $\mathfrak{G}=\mathfrak{S}/\mathfrak{R}$, where $\mathfrak{S}$ is the free group on generators S_i and $\mathfrak{R}$ the normal subgroup defined by "relations" R_j. If $\bar{\bar{\mathfrak{R}}}$ is the free operator group on generators $\bar{R}_j$ with right operators from $\mathfrak{S}$, the correspondence $\bar{R}_j{}^s\rightarrow s^{-1}R_js$ defines a homomorphism $w: \bar{\bar{\mathfrak{R}}}$ onto $\mathfrak{R}$. The kernel always contains the elements $r^{-1}qrq^{-w(r)}$ (for r, q in $\bar{\bar{\mathfrak{R}}}$). Reducing $\bar{\bar{\mathfrak{R}}}$ modulo these elements gives a group $\bar{\mathfrak{R}}$, and a homomorphism $w': \bar{\mathfrak{R}}$ onto $\mathfrak{R}$. The kernel $\mathfrak{N}_0$ of w' is the group of identities of relations.

A Σ-system Σ is a set of S_i, of $R_j=\prod S_{i_{jl}}^{\epsilon_{jl}}$, and of $J_k=\prod \bar{R}_{j_{kl}}^{\epsilon_{kl} s_{kl}}$, such that all $w'(J_k)=1$. Then Σ determines a group $\mathfrak{G}$; however, it is permitted that the identities J_k may define a proper subgroup of $\mathfrak{N}_0$. Equivalence of Σ-systems is studied by generalized Tietze and Nielsen transformations, and some invariants are found. A derived system $\Sigma\mathfrak{U}'$ is constructed by the Schreier process for each subgroup $\mathfrak{U}$ of $\mathfrak{G}$.

Each regular 3-dimensional cell complex $\mathfrak{C}$ determines a system $\Sigma_{\mathfrak{C}}$. After identification in $\mathfrak{C}$ along a maximal tree, the S_i correspond to the remaining 1-cells s_i. The R_j correspond to polygonal paths $r_j=\prod s_i{}^{\epsilon_i}$ bounding the 2-cells f_j, and the J_k to products $\prod w_j{}^{-1}r_j{}^{\epsilon_j}w_j$ representing the null path, where the r_j correspond to the faces of a 3-cell, and the w_j are Hilfswege. The system $\Sigma_{\mathfrak{C}}$ is unique within equivalence and invariant under subdivision. An analogous construction is applied to 3-manifolds in normal form (single vertex and 3-cell). To a covering $\bar{\mathfrak{C}}$ of $\mathfrak{C}$ with fundamental group $\mathfrak{U}$ corresponds the derived system $(\Sigma_{\mathfrak{C}})\mathfrak{U}'$. The Reidemeister homotopy-ring of $\mathfrak{C}$ is derived algebraically from $\Sigma_{\mathfrak{C}}$. Application of this method to knot-theory is reserved for a later paper. *R. C. Lyndon* (Princeton, N. J.).

Higman, Graham, Neumann, B. H., and Neumann, Hanna. Embedding theorems for groups. J. London Math. Soc. 24, 247–254 (1949).

If A and B are isomorphic subgroups of G, their isomorphism can be extended to an inner automorphism of a group $H\supseteq G$. The corresponding theorem holds for every infinite system of pairs A_ν, B_ν of isomorphic subgroups of G; the transforming elements can be selected as free generators of a free subgroup of H. If all the elements other than 1 of G are of infinite order, G can be extended to G^* such that in G^* all the elements other than 1 are conjugate. If G is countable, the construction furnishes a countable G^*. These theorems are proved by the method of free products with amalgamated subgroups. They are applied to prove that every countable group G can be embedded into a group H with 2 generators in such a way that the defining relations of G are in one-to-one correspondence with those of H. For set-theoretical reasons, there is no universal two-generator group over all the countable groups. *F. W. Levi.*

Hall, Marshall, Jr. Coset representations in free groups. Trans. Amer. Math. Soc. 67, 421–432 (1949).

The paper is a continuation of one by Hall and Rado [same Trans. 64, 386–408 (1948); these Rev. 10, 98]. For notations cf. that paper and its review. When a subgroup of a free group F_r is given by a standard representation $U=U[\{G\}, \phi(H)]$, the decision problem can be solved by a finite construction. If U is given by a set of generators, successive changes of 4 different types furnish a system of free generators which correspond to an "alphabetical representation" of U. These generators also admit a solution of the decision problem for U. The alphabetical representation is based on an "alphabetical ordering" of the elements $\alpha=S_i{}^\epsilon \cdots S_k{}^\eta$ (reduced form, $\epsilon, \eta=\pm1$) of F_r; the free generators of F_r and their inverses are ordered $S_1<S_1{}^{-1}<\cdots<S_r<S_r{}^{-1}$ and the elements α according

(1) their length, (2) the orders of the successive factors $S_i^{\epsilon}, \cdots, S_k^{\nu}$. When every left-coset of U is represented by its element of lowest alphabetical order, the representatives form a Schreier system $\{G\}$ and $U[\{G\}, \phi(H)]$ is the alphabetical representation of U. Necessary and sufficient conditions for a system $\alpha_1, \alpha_2, \cdots$ to be the free generators of U given by the alphabetical representation of U are established in terms of "semi-alphabetic ordering." Necessary and sufficient conditions for standard representations of normal subgroups of F_r are established. If F is an arbitrary free group, $H \subset F$ has a finite number of generators and $\beta_1, \cdots, \beta_n \epsilon F-H$, then there exists a group $\bar{H} \supseteq H$ which is finite under F and $\beta_1, \cdots, \beta_n \epsilon F-\bar{H}$. [Misprint: on p. 428, line 20, read 4.2 for 4.3.] *F. W. Levi* (Bombay).

Federer, Herbert, and Jónsson, Bjarni. Some properties of free groups. Trans. Amer. Math. Soc. **68**, 1–27 (1950).

It is proved here that the Nielsen process for finding free generators of a finitely generated subgroup of a free group will always terminate in a computable number of steps. Generalizing a result of F. Levi [Math. Z. **32**, 315–318 (1930)] the authors prove that if the elements of a subgroup H of a free group F are well ordered in any way consistent with length, then H is freely generated by those elements not expressible in terms of their predecessors. Levi had used a particular ordering. The homomorphisms of free groups onto free groups are characterized. It is shown that if the free group F is mapped homomorphically onto the free group K, then F is the free product of two groups F_1 and F_2 such that F_1 is mapped onto K isomorphically and F_2 is mapped onto the identity. In addition, if F is a finitely generated free group and $F \supset F_1$ a method is given for deciding whether or not F_1 is a free factor of F. The group F_1 must be finitely generated and in $F = F_1 * H$ it will be possible to find free generators for H whose lengths do not exceed the length of the longest generator for F_1. As an illustration of this method it is shown that the element $x^2y^3 = a$ does not generate a free factor of the free group generated by x and y. A retract of a group G onto a subgroup H is an endomorphism r of G such that $r(G) = H$, $r(x) = x$ for all $x \epsilon H$. It is noted here that though a free factor is necessarily a retract, conversely the mapping $x \to a^{-1}$, $y \to a$, where $a = x^2y^3$, yields a retract of the free group on x and y onto a subgroup which is not a free factor.
M. Hall (Columbus, Ohio).

Mal'cev, A. I. Generalized nilpotent algebras and their associated groups. Mat. Sbornik N.S. **25**(67), 347–366 (1949). (Russian)

Definitions: let G be a group with lower central series G^i; G is an N-group if $\bigcap G^i = 1$; x is a generalized periodic element if for every i there exists n with $x^n \epsilon G^i$; x has infinite p-height if for every i and n there exists y with $x \equiv y^{p^n}$ (mod G_i). A ring A is an N-ring if $\bigcap A^n = 0$. The group of quasi-regular elements of a ring A, relative to the operation $a \circ b = a + b + ab$, we shall call the QR-group of A. In § 1 there are preliminary observations, the main one being that the QR-group of an N-ring is an N-group. In § 2 the author determines necessary and sufficient conditions for an N-group G to be embeddable in the QR-group of an N-algebra: for characteristic 0, G must have no generalized periodic elements, and for characteristic p, G must have no elements of infinite p-height. In the case of characteristic p, the proof of sufficiency proceeds by constructing a suitable group algebra; in the case of characteristic 0, the author uses previous work [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 201–212 (1949); these Rev. **10**, 507] to construct a certain Lie algebra, and then passes to the associated Birkhoff-Witt algebra. These results are applied in § 3 to get conditions under which the free product of N-groups is an N-group (it being always true that the free product of N-algebras is an N-algebra). Finally the author considers the transfinite lower central series. The QR-group of a suitable algebra provides a group whose series terminates at an arbitrary ordinal; this answers problem 18 in the report of Kuroš and Černikov [Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 18–59 (1947); these Rev. **10**, 677]. A similar construction answers the same question for the derived series, settling problem 6 [erroneously cited as problem 12]. Parenthetically, it is noted that the infinite dihedral group provides a trivial counter-example to problems 29 and 31; for its derived series ends at 1 in two steps, yet it has a non-nilpotent finite homomorphic image, and its elements of finite order do not form a subgroup. *I. Kaplansky.*

Mal'cev, A. I. On ordered groups. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 473–482 (1949). (Russian)

Let G be a (simply) ordered group. Then H is a convex subgroup if $a, b \epsilon H$ and $a < c < b$ imply $c \epsilon H$. The set of convex subgroups (1) contains 1 and G, (2) is simply ordered by inclusion, (3) is closed under intersection and union, (4) is closed under conjugation, (5) if B is a convex subgroup immediately beneath A, then A/B is Abelian with no elements of finite order, (6) the inner automorphisms induced on A/B generate an integral domain which can be ordered so that the former are positive. Conversely if a group admits a system of subgroups satisfying these six postulates, it can be ordered. [In a note added in proof, the author notes that a similar theorem has been proved by Iwasawa, J. Math. Soc. Japan **1**, 1–9 (1948); these Rev. **10**, 428.] In § 2 it is shown that any two direct decompositions of an ordered group (lexicographic order on the product) have isomorphic refinements. The final section discusses the order type of an ordered group G. In particular, if G is countable, its order type is ζ^λ or $\zeta^\lambda\eta$, where λ is an ordinal and ζ, η are the order types of the integers and rational numbers. From this one sees that for any countable ordered group there is an Abelian ordered group with the same order type. Whether this is true in the uncountable case is left open. *I. Kaplansky.*

Fuchs, Ladislas. On the extension of the partial order of groups. Amer. J. Math. **72**, 191–194 (1950).

A partial order of an Abelian group G is called normal if $na \geqq 0$, with $a \epsilon G$ and n a positive integer, implies $a \geqq 0$. [It is called s-closed by P. Lorenzen, Math. Z. **45**, 533–553 (1939), p. 538; these Rev. **1**, 101.] If P and R are partial orders defined on an Abelian group G, R is called an extension of P if $a > b$ in P implies $a > b$ in R. Main theorem: for every normal partial order P defined on G, and for every pair of elements a, b of G noncomparable in P, there exists a linear extension L of P in which $a > b$. Corollary: a partial order P of an Abelian group G is the logical product of a set of linear orders L_γ of G ($a < b$ in P if and only if $a < b$ in every L_γ) if and only if P is normal. [This implies the known theorem that a partially ordered Abelian group is a subdirect sum of linearly ordered groups Γ_γ if and only if it is s-closed, and gives the additional fact that we may choose each Γ_γ to be the group G itself endowed with a linear order L_γ. The theorem stated is theorem 1 in the reviewer's

paper [Ann. of Math. (2) **41**, 465–473 (1940); these Rev. **2**, 4] and is a special case of Satz 14 in the paper of Lorenzen cited above.] Another corollary is the theorem due to F. Levi [Rend. Circ. Mat. Palermo **35**, 225–236 (1913)] that an Abelian group can be linearly ordered if and only if it contains no element other than 0 of finite order.

A. H. Clifford (Baltimore, Md.).

Everett, C. J. Note on a result of L. Fuchs on ordered groups. Amer. J. Math. **72**, 216 (1950).

A very simple proof is given of the theorem that any partial order on an Abelian group G can be extended to a linear order on G. [This is H. Shimbireva's theorem 5, Mat. Sbornik (N.S.) **20**(62), 145–178 (1947); these Rev. **8**, 563. The method is easily extended to prove Fuchs's theorem, as stated in the preceding review, by first adjoining $a-b$ to the semi-group of positive elements of G.]

A. H. Clifford (Baltimore, Md.).

Cohen, L. W., and Goffman, Casper. The topology of ordered Abelian groups. Trans. Amer. Math. Soc. **67**, 310–319 (1949).

Let G be an Abelian group (the group operation being written as addition) provided with a complete ordering $\leqq$ such that (1) $0<x$ and $0<y$ imply $0<x+y$ and (2) for every $x>0$, there exists a y such that $0<y<x$. The authors first prove that G is a topological group in its order topology, using for this purpose a directed system $\{a_\xi\}_{\xi<\xi^*}$ of elements of G, well-ordered under the group ordering $>$, having limit 0, and such that ξ^* is the least ordinal number for which such a directed system $\{a_\xi\}_{\xi<\xi^*}$ exists. The authors next introduce lower segments $X\subset G$, which are subsets of G not containing greatest elements and such that $x\epsilon X$ and $y<x$ imply that $y\epsilon X$. A lower segment X is Dedekindean if, for all $y>0$, there exists $x\epsilon X$ such that $x+y$ nonϵX. The set G_D of all Dedekindean lower segments is made an ordered group in the usual way, and it is shown that G_D is an ordered Abelian extension of G which is complete in the uniform structure of G_D. Two positive elements $x, y\epsilon G$ are said to be relatively Archimedean if for some positive integers m, n, the inequalities $y\leqq nx$ and $x\leqq my$ obtain. An ordered group $H\supset G$ is Archimedean over G if for every $h>0$ in H there is an $x\epsilon G$ such that x and h are relatively Archimedean. The group G is a-complete if it admits no proper Archimedean extensions. The authors prove that if G is a-complete, then it is complete as a topological group. The converse is false, but if G is topologically complete and dense in every Archimedean extension, then it is a-complete. The paper closes with a discussion of certain groups defined by H. Hahn [Akad. Wiss. Wien, S.-B. IIa. **116**, 601–655 (1907)]. It is proved that such groups are topologically complete, of the second ξ^*-category, and have the property that a lower segment has a least upper bound if and only if it is Dedekindean. *E. Hewitt* (Seattle, Wash.).

Halanay, A. La théorie de Galois des extensions séparables infinies et les groupes topologiques. Bull. Math. Soc. Roumaine Sci. **48**, 65–76 (1947).

The author points out that the closure operator in Krull's [Math. Ann. **100**, 687–698 (1928)] Galois theory of infinite extensions corresponds to the topology in the automorphism group defined by taking as neighborhoods of the identity element a certain class of normal subgroups of finite index. (In Krull's case these are the Galois groups of the normal extensions of finite degree.) He deduces some elementary theorems about arbitrary groups with this sort of topology.

G. Whaples (Bloomington, Ind.).

Mautner, F. I. Unitary representations of locally compact groups. I. Ann. of Math. (2) **51**, 1–25 (1950).

Le but de cet article est d'appliquer la "reduction theory" de J. von Neumann [Ann. of Math. (2) **50**, 401–485 (1949); ces Rev. **10**, 548] à la théorie des groupes discrets. Les résultats principaux sont les suivants. Soient H un espace de Hilbert séparable, $\mathfrak{F}$ une famille autoadjointe d'opérateurs dans H, $\mathfrak{P}$ une sous-algèbre autoadjointe commutative maximale de l'anneau $\mathfrak{F}'$, commutant de $\mathfrak{F}$; si l'on décompose H en somme continue (direct integral) relativement à $\mathfrak{P}$: $H=\int_\oplus H_t$, chaque $A\epsilon\mathfrak{F}$ a des composantes $A(t)$ dans les divers H_t; l'auteur démontre tout d'abord (en utilisant les théorèmes de von Neumann sur la décomposition d'un anneau en facteurs) que, quel que soit le choix des $A(t)$, ceux-ci forment une famille $\mathfrak{F}(t)$ qui, pour presque chaque t, est irréductible. Si G est un groupe discret dénombrable, $x\rightarrow U(x)$ une représentation unitaire de G, et si l'on applique ce qui précède à la famille des $U(x)$, on voit tout d'abord que, pour presque chaque t, $x\rightarrow U(x;t)$ est une représentation unitaire de G (ceci parce qu'il n'existe qu'une infinité dénombrable de relations entre les éléments de G), d'où résulte qu'on peut décomposer toute représentation de G en représentations irréductibles: on déduit de là des théorèmes analogues à ceux de Bochner et de Plancherel. Une autre généralisation du théorème de Plancherel s'obtient comme suit; soient G un groupe discret dénombrable, H l'espace des fonctions de carré sommable sur G, $\mathfrak{R}$ l'anneau d'opérateurs engendré dans H par les translations à gauche dans H; soit $H=\int_\oplus H_t$ la décomposition de H relativement au centre de $\mathfrak{R}$; $\mathfrak{R}$ se décompose alors en facteurs (au sens de Murray et von Neumann) $\mathfrak{R}(t)$, lesquels sont de classe finie; à toute fonction $f\epsilon H$ telle que l'opérateur $g\rightarrow f*g$ soit borné correspond alors dans chaque H_t un opérateur $F(t)\epsilon\mathfrak{R}(t)$; si l'on choisit convenablement une trace relative dans chaque $\mathfrak{R}(t)$, on a alors l'extension cherchée du théorème de Plancherel sous la forme suivante: pour deux fonctions f, g vérifiant les hypothèses précédentes, on a $(f,g)=\int \mathrm{Tr}\,(F(t)G(t)^*)ds(t)$, où le produit scalaire du premier membre est pris au sens de H. Dans le cas où le sous-groupe de G formé des éléments possédant un nombre fini de conjugués est fini, la somme "continue" ci-dessus ne comprend qu'un nombre fini de termes, et l'on peut associer à chaque H_t une fonction centrale χ_t jouant le rôle de "caractère," les χ_t formant de plus une base de l'ensemble des fonctions centrales de carré sommable.

En général, il n'existe aucune relation simple entre la décomposition de $\mathfrak{R}$ relativement à son centre et sa décomposition en composantes irréductibles; en particulier, on n'a rien d'analogue à la correspondance entre "caractères" et classes de représentations irréductibles; l'auteur démontre en effet ceci: si une famille autoadjointe $\mathfrak{F}$ d'opérateurs d'un espace de Hilbert H admet une décomposition en parties irréductibles et deux à deux équivalentes, alors l'anneau engendré par $\mathfrak{F}$ est, dans la terminologie de Murray et von Neumann, un facteur de classe (I); or les groupes discrets infinis conduisent en général à des facteurs de type (II_1). Il y a donc tout lieu de croire (ce qui est confirmé entièrement par les travaux plus récents) que, contrairement à ce qui se passe dans les cas classiques, la théorie des groupes doit maintenant se diviser en deux parties en

général distinctes: théorie des représentations unitaires irréductibles, et théorie des "caractères." *R. Godement.*

Godement, Roger. Sur la théorie des caractères. I. Définition et classification des caractères. C. R. Acad. Sci. Paris **229**, 967–969 (1949).

Let G be a unimodular locally compact group. By a "trace" the author understands a measure of Radon μ such that $\int(\tilde{f}*f)d\mu \geqq 0$ and $\int(f*g)d\mu = \int(g*f)d\mu$ whenever f and g are continuous functions which vanish outside of compact sets and $*$ denotes convolution with respect to Haar measure. Every trace on G defines in a natural manner a "double unitary representation" of G, that is, a pair U, V of unitary representations of G in the same Hilbert space $\mathfrak{H}$ and an involution S in $\mathfrak{H}$ such that each U_x commutes with each V_y and $V_x = SU_xS$ for all x in G. When μ is a point measure concentrated at the identity then U and V are the left and right regular representations of G. Given a trace μ let R_s and R_d denote the rings (in the sense of Murray and von Neumann) generated by the U_x and the V_x, respectively. The principal result announced in this note is that R_s and R_d are commutators of one another so that in particular the set of operators which commute with all U_x and all V_x form a commutative ring $R_s \cap R_d$.

When the double representation is irreducible in the sense that $R_s \cap R_d$ contains only multiples of the identity the generating trace is called a character. When μ is a character R_s and R_d are isomorphic factors (in the sense of Murray and von Neumann) and for G separable one may obtain a classification of characters paralleling the well-known classification of factors. The author announces that characters of classes I_n, I_∞, II_1 and II_∞ exist but that no group has a character of class III_∞. Characters of finite class (I_n or II_1) are characterized as those which are absolutely continuous with respect to Haar measure and have a continuous Radon-Nikodým derivative. Characters of class I_∞ are characterized as those arising in a certain way from a particularly manageable sort of irreducible unitary representation.

G. W. Mackey (Princeton, N. J.).

Godement, Roger. Sur la théorie des caractères. II. Mesures et groupes de classe finie. C. R. Acad. Sci. Paris **229**, 1050–1051 (1949).

The terminology is that of the preceding review. In this note the author is concerned with the representation of a general trace as a (perhaps continuous) direct sum of characters and announces such a decomposition for the special case in which only characters of finite class can arise. Specifically he defines a trace μ to be of finite class if the associated ring R_s is of finite class in the sense of Dixmier [same C. R. **228**, 152–154 (1949); these Rev. **10**, 381]. He then defines a group G to be of finite class if every trace on G is of finite class. He asserts that a group G has this property if and only if every neighborhood of the identity admits a subneighborhood which is invariant under all inner automorphisms of G. Examples include all compact groups, all discrete groups and all (locally compact) Abelian groups. To an arbitrary trace μ of finite class the author associates in a canonical fashion a certain locally compact space X whose elements are traces of finite class. He asserts that there exists a Radon measure $\hat{\mu}$ in X such that μ is in a certain precise sense the integral with respect to $\hat{\mu}$ of the members of X and that if G is separable then almost all members of X are characters. This result applied to the trace associated with the regular representations implies a Plancherel theorem with X as dual object for separable groups of finite class. *G. W. Mackey* (Princeton, N. J.).

Godement, Roger. Sur la théorie des caractères. III. Un exemple de mesure-caractère de classe (I_∞). C. R. Acad. Sci. Paris **229**, 1107–1109 (1949).

The terminology is that of the two preceding reviews. In this note the author exhibits a group for which characters not absolutely continuous with respect to Haar measure (and hence not reducible to functions) actually arise. The group is the group of all 3×3 real matrices with ones along and zeros above the main diagonal. The author obtains a Plancherel theorem for this group in which the components of the regular representation correspond to characters which are of type I_∞ and which are proper measures.

G. W. Mackey (Princeton, N. J.).

***Godement, Roger. L'analyse harmonique dans les groupes non abéliens.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, supplement, 16 pp. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

This is an expository article. Selections from the existing literature are described briefly. The case of a direct product of an Abelian with a compact group is given special consideration. A number of conjectures, open questions, and viewpoints are formulated. *I. E. Segal* (Chicago, Ill.).

Iwasawa, Kenkiti. Über nilpotente topologische Gruppen. I. Proc. Japan Acad. **21** (1945), 124–137 (1949).

Soit G un groupe topologique; l'auteur définit comme suit la série (transfinie) des commutateurs topologiques $D_\xi(G)$: on pose $D_0(G)=G$; si $\xi=\eta+1$, $D_\xi(G)$ est le sous-groupe fermé engendré par les commutateurs des éléments de $D_\eta(G)$; si ξ est un nombre limite, $D_\xi(G)$ est l'intersection des $D_\eta(G)$ avec $\eta<\xi$. De même, on définit $Z_\xi(G)$ comme étant, si $\xi=\eta+1$, le sous-groupe fermé engendré par les commutateurs des éléments de $Z_\eta(G)$, et si ξ est un nombre limite, comme étant l'intersection des $Z_\eta(G)$ avec $\eta<\xi$. Enfin, on définit la série centrale croissante $z_\xi(G)$ comme suit: $z_0(G)=e$; si $\xi=\eta+1$, $z_\xi(G)/z_\eta(G)$ est le centre de $G/z_\eta(G)$; si ξ est un nombre limite, $z_\xi(G)$ est le sous-groupe fermé engendré par les $z_\eta(G)$ avec $\eta<\xi$. Le but de cet article est avant tout d'appliquer ces notions, bien connues dans les groupes finis, aux groupes compacts. Un premier résultat est que, pour un groupe compact G, les groupes $D_\xi(G)$ (resp. $Z_\xi(G)$) coincident à partir de $\xi=\omega$, et même à partir d'un certain entier fini si G est un groupe de Lie. Par ailleurs, on dit qu'un groupe G est résoluble si pour un certain ξ on a $D_\xi(G)=e$; on dit que G est inférieurement (resp. supérieurement) nilpotent si, pour un ξ, on a $Z_\xi(G)=e$ (resp. $z_\xi(G)=G$); contrairement à ce qui se passe dans les groupes finis, ces deux dernières notions ne sont pas toujours équivalentes.

Les principaux résultats obtenus sont les suivants. La composante connexe de l'unité dans un groupe compact résoluble est abélienne; la composante connexe de l'unité dans un groupe compact inférieurement nilpotent G est contenue dans le centre de G (ces deux théorèmes se démontrent en utilisant les représentations de G, et en tenant compte du fait qu'un groupe de Lie compact, connexe et résoluble est abélien); pour qu'un groupe compact totalement discontinu soit nilpotent inférieurement, il faut et il suffit qu'il soit le produit direct de ses groupes de Sylow. Un groupe compact nilpotent supérieurement est résoluble; s'il est en outre totalement discontinu, il est aussi nilpotent inférieurement.

La fin de l'article est consacrée à étudier les groupes de Lie compacts nilpotents supérieurement (d'après ce qui précède, un tel groupe G est l'extension d'un tore de dimension finie par un groupe fini); appelons classe d'un tel groupe le nombre transfini ξ tel que l'on ait $z_\eta(G)\neq G$ pour $\eta<\xi$, $z_\xi(G)=G$; alors l'auteur démontre que la classe d'un groupe de Lie compact nilpotent supérieurement peut prendre les valeurs $1, 2, \cdots, \omega+1, \omega+2, \cdots$ seulement, et donne un exemple d'un groupe de classe $\omega+n$. Le point essentiel est de prouver que, pour un groupe de Lie compact nilpotent supérieurement G, on a toujours $z_\omega(G)\supset G_0$, où G_0 est la composante connexe de l'unité dans G, et $z_{\omega+n}(G)=G$ pour un certain entier n; la démonstration de cette propriété utilise naturellement la structure des groupes finis nilpotents (produit direct d'un p-groupe de Sylow par un groupe d'ordre premier à p). *R. Godement* (Nancy).

Mostow, George Daniel. A new proof of E. Cartan's theorem on the topology of semi-simple groups. Bull. Amer. Math. Soc. **55**, 969–980 (1949).

Le théorème en question est le suivant: tout groupe de Lie semi-simple connexe est homéomorphe au produit direct d'un groupe compact et d'un espace euclidien. L'auteur prouve d'abord le théorème suivant (qui avait été vérifié par É. Cartan depuis longtemps, mais dont, apparemment, on ne connaissait pas encore de démonstration directe): soit L une algèbre de Lie semi-simple sur le corps réel; soient L_C l'algèbre déduite de L par passage du réel au complexe, et θ l'involution de L_C définie par $\theta(X+iY)=X-iY$ $(X, Y\epsilon L)$; alors il existe une sous-algèbre L_K de L_C qui est invariante globalement par θ, qui est "compacte" (i.e., sa forme fondamentale est définie), et qui est une forme réelle de L_C. Le théorème de Cartan se déduit de là par une série de lemmes. Soient G un groupe semi-simple connexe, L son algèbre de Lie; soient L_C, L_K comme ci-dessus; on a tout d'abord la relation $L=L\cap L_K\oplus L\cap iL_K$; d'autre part, soient Aut (L_C) le groupe des automorphismes de L_C, Aut (L) et Aut (L_K) les sous-groupes de Aut (L_C) qui conservent respectivement L et L_K; l'auteur montre tout d'abord que ces groupes de matrices sont algébriques (sur le corps réel); si l'on note G_C^*, G^* et G_K^* les composantes connexes de l'unité dans les trois groupes précédents, on vérifie facilement (du fait que toute dérivation d'une algèbre de Lie semi-simple est intérieure) que G^*, par exemple, est le groupe engendré par les dérivations de L_C associées aux éléments de L, des propriétés analogues ayant lieu pour G_C^* et G_K^*; maintenant, considérons $J=L\cap L_K$ et $S=L\cap iL_K$; les dérivations de L_C définies par les éléments de J engendrent un sous-groupe F^* de G_C^*; soit par ailleurs S^* l'ensemble des éléments de G_C^* de la forme exp (X) où X est une dérivation associée à un élément de S; puisque L_K est "compacte," on peut choisir une base de L_C relativement à laquelle les éléments de G_K^* sont des matrices orthogonales réelles; on constate tout d'abord que F^* est le sous-groupe de G^* formé des matrices réelles orthogonales; si alors, pour un $g\epsilon G^*$, on écrit la décomposition canonique $g=f\cdot s$ (f: matrice orthogonale réelle; $s=\exp(iW)$ où W est réelle antisymétrique), on démontre, en utilisant le fait que Aut (L) est algébrique, que $f\epsilon F^*$ et $s\epsilon S^*$; de là résulte que G^* est homéomorphe au produit $F^*\times S^*$ d'un groupe compact par un espace euclidien; il reste à passer de là au cas de G, ce qui se fait en observant que G^* est le groupe adjoint de G, donc que $G^*=G/D$ où D est le centre-discret de G. [Un théorème plus général (relatif aux groupes de Lie quelconques) a été prouvé récemment, par des méthodes un peu différentes; voir Iwasawa, Ann. of Math. (2) **50**, 507–558 (1949); ces Rev. **10**, 679.]

R. Godement (Nancy).

Leray, Jean. Sur l'anneau de cohomologie des espaces homogènes. C. R. Acad. Sci. Paris **229**, 281–283 (1949).

(1) Let X be a compact simply connected Lie group, Y a closed "nonexceptional" Lie subgroup. Let T be a maximal toroid of Y. The reciprocal of the natural mapping $U=X/T\rightarrow W=X/Y$ is an isomorphism of H_W, the cohomology algebra of W (over a field of characteristic zero) onto the totality of elements of H_U which are invariant under a certain finite group operating on H_U. (2) Let X be a simple nonexceptional compact Lie group of rank l, R a maximal toroid of X, S a singular toroid of X of rank $l-1$. Then $H_{X/S}=H''\otimes\Lambda H'$ where H'' is $H_{X/R}$ modulo the ideal generated by certain elements of order 2 and $\Lambda H'$ is the exterior algebra of a certain module H' of rank 1. (3) Let ξ be the projection of a fibred space X of dimension D onto its base Y of dimension $D-d$. Assume that the fibre F has the cohomology algebra of a d-sphere and that X, Y, F are compact and orientable. The author displays formulas for the Poincaré polynomials of H_X and H_Y as functions of the polynomial of $\xi^{-1}H_Y$, and describes the properties of certain natural mappings associated with the situation. Necessary and sufficient conditions are given for the existence of a module P such that $H_X=\xi^{-1}H_Y\otimes\Lambda P$.

P. A. Smith (New York, N. Y.).

Wang, Hsien-Chung. Homogeneous spaces with non-vanishing Euler characteristics. Ann. of Math. (2) **50**, 925–953 (1949).

L'auteur étudie les espaces homogènes G/L à groupe compact de Lie G et à caractéristique d'Euler $\chi\neq0$. D'après Hopf et Samelson le sous-groupe L doit contenir un groupe maximal abélien de G, et dans ce cas on a la relation $\chi(G)=\chi(L)\chi(G/L)$ entre les caractéristiques. L'auteur réduit le problème de la détermination de tous ces espaces au cas d'un G simple. En se bornant aux quatre grandes classes, il exprime les sous-groupes L possibles comme des produits directs d'autres groupes (non exceptionnels), d'abord localement, puis globalement (en se servant d'un théorème de Malcev assurant l'existence globale de ces sous-groupes). Les L peuvent être obtenus comme les sous-groupes maximaux de G qui laissent invariantes un certain nombre de variétés linéaires, parmi lesquelles il peut y avoir (dans le cas des groupes orthogonaux) des couples complexes conjuguées. L'auteur calcule pour les grandes classes les caractéristiques d'Euler et quelques groupes d'homotopie de G/L. [Les espaces G/L ont également été étudiés par J. Leray [C. R. Acad. Sci. Paris **228**, 1545–1547, 1784–1786, 1902–1904; ces Rev. **10**, 680; **11**, 10, 80].] *H. Freudenthal.*

Borel, A., et De Siebenthal, J. Les sous-groupes fermés de rang maximum des groupes de Lie clos. Comment. Math. Helv. **23**, 200–221 (1949).

Let G be a compact Lie group, G' a closed connected subgroup having the same rank as G. Let Z' be the center of G'. The main object of this paper is to show that G' is a connected component of the normalizator of Z' in G. To prove this the authors employ the vector diagram of G, first showing that the root vectors of G' are also root vectors of G (this is not true for arbitrary subgroups G'). The result in question is a consequence of a proposition which gives a necessary and sufficient condition that a subsystem of root

vectors of G be the root vectors of a closed subgroup of G. There is included an explicit determination of the subgroups G' for the simple groups G. *P. A. Smith.*

Maak, Wilhelm. Moduln fastperiodischer Funktionen. Abh. Math. Sem. Univ. Hamburg **16**, 56–71 (1949).

The author proves the main theorem that every closed left-module of almost periodic functions on a group is the sum of irreducible finite-dimensional left-modules. Closure is formed with respect to uniform convergence, and a nonvoid set of almost periodic functions is called a left-module if the set together with f and g contains $\alpha f+\beta g$ (α and β constants) and together with $f(x)$ contains $f(cx)$. The sum appearing in the theorem is formed by taking finite sums of functions from the components and next forming the closure. Thus, in particular, the approximation theorem for almost periodic functions is proved. The proof differs from known proofs by aiming directly at these theorems which are reached without recourse to the theory of Fourier series (unicity theorem, Parseval equation, etc.). The main theorem is well suited for application to spherical harmonics.
E. Følner (Copenhagen).

Clifford, A. H. Semigroups without nilpotent ideals. Amer. J. Math. **71**, 834–844 (1949).

The opening sections of this paper are concerned with semi-groups containing a zero element 0, but no proper nilpotent ideals, left or right (an ideal B is nilpotent if $B^r=0$ for some r). The structure of a minimal non-zero two-sided ideal M of such a semi-group S is investigated. This is necessarily simple, and if it satisfies the further restriction that it contains both a minimal left ideal and a minimal right ideal, is completely simple, in the sense defined by the reviewer [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127]. The case where S is simple implies a result, due to the reviewer, to the effect that a simple semi-group is completely simple if and only if it contains a primitive idempotent [Proc. Cambridge Philos. Soc. **37**, 434–435 (1941); these Rev. **3**, 199]. The case where S above satisfies the restriction that every two-sided ideal of S contains minimal left and right ideals is next considered. It is proved that every left, right or two-sided ideal of S contains, respectively, a minimal left, right or two-sided ideal. Further, every minimal left ideal of S is contained in some minimal two-sided ideal of S. Finally it is proved that every left ideal L of S contains a nonzero idempotent e, and that, if L is minimal, $L=Se$.

The final section is concerned with the relation between these results and Schwarz's theory of semi-groups with a kernel N and a radical R (N is the (possibly zero) minimal two-sided ideal of S, R is the maximal ideal of S such that $R^r\subset N$, if such a maximal ideal exists) [see Sborník Prác Prírodovedeckej Fakulty Slovenskej Univerzity v Bratislave no. **6** (1943); these Rev. **10**, 12]. It is proved that if S possesses a radical R and, in addition, satisfies the descending chain condition for left and right ideals, then every left (or right) ideal of S not contained in R contains an idempotent not in R. *D. Rees* (Cambridge, England).

Rich, R. P. Completely simple ideals of a semigroup. Amer. J. Math. **71**, 883–885 (1949).

This paper is complementary to a paper by Clifford [see the preceding review]. It contains three results, the first of which is, essentially, the same as that of the first part of the paper cited above, but under very much weaker hypotheses. This result is that, if S is a semi-group with zero, and M is a minimal two-sided ideal of S which contains both a minimal left ideal and a minimal right ideal, and in addition, satisfies the restriction that $M^2\neq0$, then M is completely simple. The other two results, which are converse to each other, are to the effect that (I) if L, R are, respectively, minimal left and right ideals of S such that both $LR\neq0$ and $RL\neq0$, then LR is completely simple; (II) if S contains a completely simple two-sided ideal M, then $M=LR$, where L, R are, respectively, minimal left and right ideals of S such that $RL\neq0$. The paper concludes with examples showing the independence of the conditions $LR\neq0$ and $RL\neq0$ in (I) above. *D. Rees* (Cambridge, England).

Tamari, Dov. Les images homomorphes des groupoïdes de Brandt et l'immersion des semi-groupes. C. R. Acad. Sci. Paris **229**, 1291–1293 (1949).

The set $E\times E$ of all pairs (k, l) of an arbitrary set E becomes a Brandt groupoid $\mathfrak{B}$ if we define (k, l) and (m, n) to be non-composable if $l\neq m$, and $(k, l)(m, n)=(k, n)$ if $l=m$. If a semi-group (associative system with cancellation) is a homomorphic image of such a $\mathfrak{B}$, it is a group; and every group G is a homomorphic image of such a $\mathfrak{B}$, namely of $G\times G$ under the mapping $(g_1, g_2)\rightarrow g_1g_2^{-1}$. It is stated without proof that a semi-group can be imbedded in a group if and only if it is the homomorphic image of a part of some $\mathfrak{B}$. Generalization to higher dimension is discussed briefly. *A. H. Clifford* (Baltimore, Md.).

Evans, T. Homomorphisms of non-associative systems. J. London Math. Soc. **24**, 254–260 (1949).

Postulates for a loop are given in terms of three binary operations: multiplication, left division and right division. Examples are constructed to show that if a mapping is a homomorphism with respect to two of the three operations (multiplication and right division, or left and right division) the image of a loop is not necessarily a loop. If, however, a homomorphism preserves all three operations the image of a loop is again a loop. The construction given makes use of the extension chains of Bates and Kiokemeister [Bull. Amer. Math. Soc. **54**, 1180–1185 (1948); these Rev. **10**, 353].
D. C. Murdoch (Vancouver, B. C.).

Prenowitz, Walter. Spherical geometries and multigroups. Canadian J. Math. **2**, 100–119 (1950).

This is the third of the author's series characterizing geometries as multigroups. [For the first two cf. Amer. J. Math. **65**, 235–256 (1943); Trans. Amer. Math. Soc. **59**, 333–380 (1946); these Rev. **4**, 251; **7**, 375.] In this paper Kline's postulates for a spherical geometry [Ann. of Math. (2) **18**, 31–44 (1916)] are shown to be in a sense equivalent to a regular commutative additive multigroup with the special properties that: (a) for each element b there exists an inverse b^*, such that $a+b^*$ contains all x for which $a\epsilon b+x$, (b) every element is idempotent, (c) the submultigroup generated by a single element (except the identity) has exactly three elements. Some of the algebraic properties of this kind of multigroup are developed.
H. H. Campaigne (Washington, D. C.).

NUMBER THEORY

Klee, V. L., Jr. A note on Fermat's congruence. Amer. Math. Monthly **56**, 626–628 (1949).

Carmichael [The Theory of Numbers, Wiley, New York, 1914, pp. 53, 54, 72] defined $\lambda(n)$ by $\lambda(2^e)=\varphi(2^e)$ if $0\leqq e\leqq 2$ (φ being Euler's function); $\lambda(2^e)=\frac{1}{2}\varphi(2^e)$ if $e>2$; $\lambda(p^e)=\varphi(p^e)$ if p is an odd prime; $\lambda(2^e p_1^{e_1} p_2^{e_2}\cdots)=$ the least common multiple of $\lambda(2^e)$, $\lambda(p_1^{e_1})$, $\cdots$, where 2, p_1, p_2, $\cdots$ are different primes. He proved that for every m and every a prime to m, $a^{\lambda(m)}\equiv 1 \pmod m$ and also that for every m there exists a g satisfying $g^x\equiv 1 \pmod m$ for $x=\lambda(m)$ but not for $x<\lambda(m)$. The author considers the set $L(m)$ of all exponents k such that, for every a prime to m, $a^k\equiv 1 \pmod m$, and proves that $L(m)$ consists of all numbers divisible by $\lambda(m)$. Let $C(n)$ denote the set of all integers h so that, for every a prime to h, $a^n\equiv 1 \pmod h$; then $n\epsilon L(m)$ if and only if $m\epsilon C(n)$; $C(n)$ consists of all divisors of $\gamma(n)$, the greatest integer in $C(n)$; $m|\gamma(n)$ if and only if $\lambda(m)|n$; $\gamma(n)=2$ if n is odd; $\gamma(n)=24v$ if n is even. Some remarks on a function $M(n)$ of Carmichael are added. [Corrections: p. 627, end of line 15: for $\lambda(n)$ read $\gamma(n)$; line 18: for m read n; p. 628, line 2: for n read $\gamma(n)$.] *N. G. W. H. Beeger.*

Pizá, Pedro A. Nombres parfaits et sommes de puissances. Mathesis **58**, 337–342 (1950).

Let $N_p=2^{p-1}(2^p-1)$, so that N_p is a perfect number in case 2^p-1 is a prime. By means of a simple algebraic identity the author expresses N_p as a sum of powers of integers in various ways, of which the following is the simplest: $N_p=\sum_{b=1}^{[\frac{1}{2}(p+1)]}\binom{p}{2b-1}\cdot 3^{p+1-2b}$. By means of one of these relations he gives a transformation of the Fermat equation $x^p+y^p=z^p$ which he hopes may be of service in proving Fermat's last theorem. *H. W. Brinkmann.*

Jarden, Dov. Arithmetical properties of sums of powers. Amer. Math. Monthly **56**, 457–461 (1949).

The first part of this paper proves the following extension of a result of Jänichen [S.-B. Berlin. Math. Ges. **20**, 23–29 (1921)]. Given an ordered set of n integers $S(1)$, $S(2)$, $\cdots$, $S(n)$, then the necessary and sufficient condition that there exists a polynomial

$$p(x)=x^n+a_1x^{n-1}+\cdots+a_n=\prod_{r=1}^{n}(x-x_r)$$

with integer coefficients such that the sums

$$s_q(p)=x_1^q+\cdots+x_n^q$$

are equal to $S(q)$, $q=1, 2, \cdots, n$, is that

$$\sum_{d|m} f(d)S(m/d)\equiv 0 \pmod m, \qquad m=1, 2, \cdots, n,$$

f being an integer-valued function satisfying the condition

$$f(1)=\pm 1, \quad \sum_{d|m} f(d)\equiv 0 \pmod m, \qquad m=1, 2, \cdots, n.$$

In Jänichen's criterion the function $f(n)$ is the Mobius function $\mu(n)$. The second part of the paper gives some congruences involving the sums $s_q(p)$. *W. H. Simons.*

Schwarz, Štefan. On the reducibility of binomial congruences and on the bound of the least integer belonging to a given exponent mod p. Časopis Pěst. Mat. Fys. **74**, 1–16 (1949). (English. Czech summary)

The author considers the factorization of x^n-a into irreducible polynomials (mod p), and gives an explicit formula for the number of irreducible polynomial factors of degree k. Some applications of the formula are made, and generalizations to the finite field $[p^f]$ are indicated. The author generalizes a result of Vinogradov by proving that if l divides $p-1$, the least integer (mod p) which belongs to the exponent l is $O(l^{s-1}p^{\frac{1}{s}})$. He generalizes similarly a result due to the reviewer [Quart. J. Math., Oxford Ser. **8**, 308–312 (1937)]. *H. Davenport* (London).

Persson, Bengt. On a Diophantine equation in two unknowns. Ark. Mat. **1**, 45–57 (1949).

It was proved by T. Nagell [Norsk Mat. Forenings Skr. (I) no. **2** (1921)] that the equation (1) $x^2+x+1=y^q$ has only trivial integral solutions unless q is a power of 3. W. Ljunggren [Acta Math. **75**, 1–21 (1943); these Rev. **8**, 135] proved that the only nontrivial solution of (1) is $y=7$ for $q=3$. The present paper discusses the equation (2) $x^2+x+\frac{1}{4}(D+1)=y^q$, where q is an odd prime, $D\equiv 3 \pmod 4$, $D\geqq 7$, D square-free. Let h denote the number of ideal classes in the field $K(\sqrt{(-D)})$. The following results are proved. If $q\nmid h$, there is only a finite number of values of D for which (2) is solvable, and any solution must satisfy the inequality $y<1+\frac{1}{4}D\operatorname{cosec}^2\pi/q$. If (2) has a solution with $y<\frac{1}{4}(D+1)$, then $q|h$. If $q=D$, (2) is not solvable. If $q=3$ and $3\nmid h$, then (2) has only the solution $y=\frac{1}{3}(D-1)$ if $D=3c^2+4$, only the solution $y=\frac{1}{3}(D+1)$ if $D=3c^2-4$, and no solutions for other values of D. If $q=3$ and $3|h$, there are other solutions: the examples $D=23$, 31, 59, 85 are discussed. If $q=5$ and $5\nmid h$, then (2) has only the solutions $D=7$, $y=2$ and $D=11$, $y=3$. If $q=7$ and $7\nmid h$, then (2) has at most one solution y for a given D. A table of solutions of (2) is given for prime $D<100$ and $q=3$. *I. Niven* (Eugene, Ore.).

Georgikopoulos, C. Rational integral solutions of the equations $x^3+4y^3=z^2$ and $x^3+2y^3=z^2$. Bull. Soc. Math. Grèce **24**, 13–19 (1948). (English. Greek summary)

It is proved that all the rational integral solutions of the equation

$$x^3+4y^3=z^2, \tag{1}$$

for x, y, z, relatively prime by pairs, are included in the parametric solution

$$\pm x=x_1(x_1^3+4y_1^3),$$
$$\pm y=y_1(y_1^3-2x_1^3),$$
$$\pm z=x_1^6-2y_1^6+10x_1^3y_1^3.$$

A similar result is worked out for the equation

$$x^3+2y^3=z^2. \tag{2}$$

The method used depends on (a) the fact that the class number of $K(\sqrt[3]{2})$ (K the field of rational numbers) is 1, (b) that 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$, form a minimal basis for $K(\sqrt[3]{2})$, and (c) that $\sqrt[3]{2}-1$ is a fundamental unit. Knowing this, one is completely informed concerning the arithmetic of $K(\sqrt[3]{2})$, and the results are obtained by comparing the factorizations into integers of $K(\sqrt[3]{2})$ of both sides of (1), (2), respectively. *H. N. Shapiro* (New York, N. Y.).

Georgikopoulos, C. On the equation $ax^2+by^2+cz^2=0$. Bull. Soc. Math. Grèce **24**, 20–25 (1948). (English. Greek summary)

The problem is considered once again of finding all solutions of the Diophantine equation of the title, given any one solution [cf. Novikov, Doklady Akad. Nauk SSSR (N.S.) **61**, 205–206 (1948); these Rev. **10**, 13]. Several

methods of achieving this are discussed, and several parametric formulae including all integral solutions of (*) are given. *H. N. Shapiro* (New York, N. Y.).

Sastry, S., and Rai, T. On equal sums of like powers. Math. Student 16 (1948), 18–19 (1949).

If a nontrivial solution in positive integers of the k equations

$$x_1^h+\cdots+x_j^h=y_1^h+\cdots+y_j^h, \qquad 1\leqq h\leqq k,$$

is denoted by

$$(1)\qquad [a_1, a_2, \cdots, a_j]_k=[b_1, b_2, \cdots, b_j]_k,$$

then it is known that the existence of a solution (1) implies the existence of solutions

$$(2)\qquad [a_1+z, \cdots, a_j+z]_k=[b_1+z, \cdots, b_j+z]_k$$

and

$$(3)\quad [a_1, \cdots, a_j, b_1+d, \cdots, b_j+d]_{k+1} =[b_1, \cdots, b_j, a_1+d, \cdots, a_j+d]_{k+1},$$

z and d being arbitrary integers. Burchnall and Chaundy have given a one-parameter solution of (1) when $k=7$ [Quart. J. Math., Oxford Ser. 8, 119–130 (1937)]. Applying the results (2) and (3) to this solution, with suitable choice of z and d, the authors have obtained a one-parameter solution of the equation

$$x_1^7+\cdots+x_8^7=y_1^7+\cdots+y_8^7.$$

[Correction: page 19 line 1 should read, ". . . that (4) implies . . .". Line 4 should read, "From (5) and (6) . . .".] *W. H. Simons* (Vancouver, B. C.).

Zahlen, J. P. Sur les égalités multigrades en nombres tous premiers. Euclides, Madrid 9, 283–286 (1949).

The following theorems and their converses are proved. If a_i and b_j are primes such that

$$(1)\qquad a_1, a_2, \cdots, a_k \overset{n}{=} b_1, b_2, \cdots, b_k \qquad (n\geqq k+1),$$

then for all $1\leqq r\leqq n$

$$(2)\quad \int(a_1)^r+\int(a_2)^r+\cdots+\int(a_k)^r=\int(b_1)^r+\cdots+\int(b_k)^r,$$

where the notation $\int(a_i)^r$, attributed to Euler, denotes the sum of the divisors of a_i^r. Moreover the existence of a multigrade equation (1) implies the existence of

$$(3)\quad \left(\int a_1\right), \cdots, \left(\int a_k\right) \overset{n}{=} \left(\int b_1\right), \cdots, \left(\int b_k\right),$$

and conversely (3) implies (1). The result (2) is obtained by adding successively the equations of (1) with $n=1, 2, \cdots$ and using the fact that if p is a prime then $\int p^r=1+p+p^2+\cdots+p^r$. The result (3) is obtained by applying the theorem of Frolov to (1). The author concludes the paper with some unanswered problems. *W. H. Simons* (Vancouver, B. C.).

Gloden, A. Analyse diophantienne et analyse multigrade. Euclides, Madrid 9, 329–331 (1949).

Solution of $ab(a+b)=cd(c+d)$ by substitution of $a=pq$, $b=rs$, $a+b=tu$, $c=ps$, $d=qt$, $c+d=ru$. Every solution also satisfies $(a+b)^k+(-a)^k+(-b)^k=(c+d)^k+(-c)^k+(-d)^k$ for $k=1, 3$. Applying a theorem [Mehrgradige Gleichungen, Noordhoff, Groningen, 1944, p. 22, Satz V; these Rev. 8, 441] to this equality, we get a solution of $\sum_{i=1}^6 A_i^k=\sum_{i=1}^6 B_i^k$, $k=1, 2, 3, 4$. *N. G. W. H. Beeger* (Amsterdam).

Nyberg, Michael. Culminating and almost culminating continued fractions. Norsk Mat. Tidsskr. 31, 95–99 (1949). (Norwegian)

The author shows that the simple continued fraction expansion of $\sqrt{D}$, D a positive integer, is almost-culminating in the following cases: (1) $D=[m^n-\frac{1}{2}(m-1)]^2+m$; number of terms in a period $=2(3n-1)$, (2) $D=[m^n+\frac{1}{2}(m-1)]^2+m$; number of terms in a period $=6n$. Here n is a positive integer and $m>1$ an odd integer. *T. Nagell* (Uppsala).

Bateman, P. T. Note on the coefficients of the cyclotomic polynomial. Bull. Amer. Math. Soc. 55, 1180–1181 (1949).

The author proves that if A_n is the largest coefficient in absolute value of the nth cyclotomic polynomial $\prod_{d|n}(1-x^d)^{\mu(n/d)}$, then for some constant c_{14} and all n, (B): $A_n<\exp\{n^{(c_{14}/\log\log n)}\}$. This was stated by Erdös [same Bull. 52, 179–184 (1946); these Rev. 7, 242] along with its implication that his conjecture (A): $A_n>\exp\{n^{(c_{13}/\log\log n)}\}$, for some constant c_{13} and infinitely many n, would be a "best" result. The inequalities (A) and (B) are misprinted in Erdős' paper, p. 182. *R. Hull* (Lafayette, Ind.).

Venkataraman, C. S. On Von Sterneck-Ramanujan function. J. Indian Math. Soc. (N.S.) 13, 65–72 (1949).

Continuation of previous work [same J. (N.S.) 10, 57–61 (1946); these Rev. 9, 225]. *N. G. de Bruijn* (Delft).

Hua, Loo-Keng, and Vandiver, H. S. On the number of solutions of some trinomial equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 35, 477–481 (1949).

The authors investigate the number N of solutions of the equation $c_1x^{a_1}+c_2x^{a_2}+c_3=0$, where c_1, c_2, c_3 are given nonzero elements in a finite field $F(p^n)$ and x_1, x_2 are also nonzero elements in that field. Inequalities are given for N that are closer than those given previously and in certain special cases (which involve, for example, that n be even) reasonably simple exact expressions for N are given. *H. W. Brinkmann* (Swarthmore, Pa.).

Hua, L. K., and Vandiver, H. S. On the nature of the solutions of certain equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 35, 481–487 (1949).

The authors investigate the number N of solutions of the equation $c_1x_1^{a_1}+c_2x_2^{a_2}+\cdots+c_sx_s^{a_s}+c_{s+1}=0$, where c_i are given elements in a finite field $F(p^n)$ and the x_i are also in that field, with $c_1c_2\cdots c_sx_1x_2\cdots x_s\neq 0$, furthermore $s\geqq 2$ and $c_{s+1}\neq 0$ when $s=2$; finally $0<a_i<p^n-1$. In the case $s=2$ an expression is derived for N in terms of generalized characters of the multiplicative group of $F(p^n)$. In the special case where $c_{s+1}=0$ (so that $s>2$) and the numbers (a_i, p^n-1) are relatively prime in pairs, the following simple expression for N is obtained:

$$N=(p^n-1)((p^n-1)^{s-1}+(-1)^s)/p^n.$$

The authors also remark that the case $s=1$ would be totally different since it is essentially equivalent to the case of power residues and it is well known that the results here are different in character from those obtained for $s\geqq 2$. *H. W. Brinkmann* (Swarthmore, Pa.).

Vandiver, H. S. Quadratic relations involving the numbers of solutions of certain types of equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 35, 681–685 (1949).

As in previous papers, the author considers the number of solutions of the equation $c_1x_1^m+c_2x_2^m+c_3=0$, where c_1, c_2, c_3

are given nonzero elements in a finite field $F(p^n)$ and x_1, x_2 are also nonzero elements in that field; it is also assumed that $0<m_1, m_2<p^n-1$ and that p^n-1 is divisible by m_1, m_2. If we set $c_1/c_3=g^i$, $c_2/c_3=g^j$, where g is a primitive root of $F(p^n)$, then the number of distinct sets $x_1^{m_1}, x_2^{m_2}$ satisfying the equation is denoted by (i, j). With this notation, an explicit expression in terms of p and n for the quadratic expression $\sum_{j=0,\cdots,m_2-1}^{i=0,\cdots,m_1-1}(i, j)\,(i+h, j+k)$ is obtained. In the special case $m_1=m_2$ the author had accomplished this in a previous paper [same Proc. 33, 236–242 (1947); these Rev. 9, 9]. *H. W. Brinkmann* (Swarthmore, Pa.).

Vandiver, H. S. On the use of indeterminates in the theory of exponential sums. Proc. Nat. Acad. Sci. U. S. A. 35, 686–690 (1949).

Let α be a primitive mth root of unity and let $F(x)$ be the irreducible polynomial of degree $\varphi(m)$ with rational integral coefficients of which α is a root. The author starts with the obvious remark that any equation of the form $g(\alpha)=0$, where $g(x)$ is a polynomial with rational integral coefficients, is equivalent to the identity $g(x)\equiv 0 \pmod{F(x)}$. He then points out some examples in which this identity can be replaced by the congruence $g(x)\equiv 0 \pmod{G(x)}$, where $G(x)=x^{m-1}+x^{m-2}+\cdots+x+1$. New congruences of this kind can then be obtained by differentiating with respect to x and similar devices. The substitution of special values for x in the stronger congruence often leads to new numerical results. *H. W. Brinkmann* (Swarthmore, Pa.).

Vandiver, H. S. On a generalization of a Jacobi exponential sum associated with cyclotomy. Proc. Nat. Acad. Sci. U. S. A. 36, 144–151 (1950).

Let $F(p^n)$ be a finite field; let $m_1, \cdots, m_s$ be factors of p^n-1; define $\operatorname{ind} a$ for $a\neq 0$ by $g^{\operatorname{ind} a}=a$, where a is in $F(p^n)$ and g is a primitive root of that field; finally set $\alpha_i=\exp(2\pi i/m_i)$. Then the author sets

$$\psi=\psi(\alpha_1^{\mu_1}, \cdots, \alpha_s^{\mu_s})=\sum \alpha_s^{\mu_s \operatorname{ind} A}\prod_{i=1}^{s-1}\alpha_i^{\mu_i \operatorname{ind} a_i},$$

where, in the summation, each a_i ranges independently over the elements of $F(p^n)$ and we have set $A=1-\sum_{i=1}^{s-1}a_i$; as a matter of definition we have also set $\alpha_i^{\mu_i \operatorname{ind} 0}=0$. The number ψ thus defined is called a generalized Jacobi sum and certain properties of it are derived in this paper. It can be expressed in terms of the characters of the multiplicative group of $F(p^n)$ and can be used in connection with the determination of the number of solutions of an equation of the type $c_1x_1^{e_1}+\cdots+c_sx_s^{e_s}+c_{s+1}=0$, where the c_i and the x_i are in $F(p^n)$ [see Hua and Vandiver, same Proc. 35, 94–99 (1949); these Rev. 10, 515]. *H. W. Brinkmann.*

✱Siegel, Carl Ludwig. Transcendental Numbers. Annals of Mathematics Studies, no. 16. Princeton University Press, Princeton, N. J., 1949. viii+102 pp. \$2.00.

"This booklet reproduces with slight changes a course of lectures delivered in Princeton during the spring term 1946. It would be misleading to call it a theory of transcendental numbers, our knowledge concerning transcendental numbers being narrowly restricted. The text deals with a few special transcendency problems of some interest, but it is more than a mere collection of scattered examples, since it involves a method which might be useful in the search of more general results."

The first chapter begins with simple proofs of the irrationality of e and π. Their transcendency, and the general theorem of Lindemann-Weierstrass, is proved in a way somewhat similar to that in the paper of K. Mahler [J. Reine Angew. Math. 166, 118–136 (1931)], using Hermite's polynomials $P_1(x), \cdots, P_m(x)$ of given degrees for which $P_1(x)e^{\rho_1 x}+\cdots+P_m(x)e^{\rho_m x}\neq 0$ has a zero of maximal order at $x=0$.

The second chapter deals with systems of differential equations

$$(1)\qquad y_k'=\sum_{l=1}^{m}Q_{kl}(x)y_l, \qquad k=1, \cdots, m,$$

where the Q's are rational functions with algebraic coefficients. Assume the matrix $Q=(Q_{kl})_{k,l=1,\cdots,m}$ consists of r square boxes $Q_t=(Q_{kl,t})_{k,l=1,\cdots,m_t}$, $1\leqq t\leqq r$, along the diagonal of Q, the elements of Q outside the boxes being zero, while $\sum_{t=1}^{r}m_t=m$. Thus (1) splits into r separate systems

$$(2)\qquad y'_{k,t}=\sum_{l=1}^{m_t}Q_{kl,t}(x)y_{l,t}, \qquad k=1, \cdots, m_t;\ 1\leqq t\leqq r.$$

Then there is a matrix $Y=(y_{kl}(x))_{k,l=1,\cdots,m}$, the columns of which form m systems of independent solutions of (1), such that Y splits into r boxes $Y_t=(y_{kl,t}(x))_{k,l=1,\cdots,m_t}$, $1\leqq t\leqq r$, of the same orders as the boxes of Q, the columns of Y_t forming now m_t systems of independent solutions of (2). The boxes Y_t are called independent if

$$\sum_{t=1}^{r}\sum_{k=1}^{m_t}\sum_{l=1}^{m_t}P^*_{k,t}(x)c_{l,t}y_{kl,t}(x)\neq 0$$

for arbitrary polynomials $P^*_{k,t}(x)$ and arbitrary constants $c_{l,t}$, unless all products $P^*_{k,t}(x)c_{l,t}$ vanish simultaneously. A series $f(x)=\sum_{n=0}^{\infty}c_nx^n/n!$ is said to be of type E if (a) all c_n lie in the same algebraic field K of finite degree over the rational field; (b) if $\epsilon>0$, then all c_n and their conjugates with respect to K are $O(n^{\epsilon n})$; (c) moreover, there exists a sequence $q_0, q_1, \cdots$ of positive integers such that c_kq_n $(0\leqq k\leqq n)$ is integral in K and that $q_n=O(n^{\epsilon n})$. We also say that the functions $E_1(x), \cdots, E_m(x)$ of type E form a normal system if (A) $y_1=E_1(x), \cdots, y_m=E_m(x)$ is a solution of a system of differential equations (1) as defined above, and (B) if (1) has a solution matrix Y consisting of independent boxes Y_t. The following lemma is then proved. Let $E_1(x), \cdots, E_m(x)$ form a normal system; let further the number $\alpha\neq 0$ and the Taylor coefficients of $E_1(x), \cdots, E_m(x)$ all lie in the same field K of finite degree h over the rational field. If α is different from the poles of all functions $Q_{kl}(x)$ $(k, l=1, \cdots, m)$, then at least $\frac{1}{2}m/h$ of the numbers $E_1(\alpha), \cdots, E_m(\alpha)$ are linearly independent over K. From this result, one easily deduces the following theorem. Let the E-functions $y=E_1(x), \cdots, y=E_m(x)$ satisfy a system of differential equations (1) as defined above, and let, for $\nu=1, 2, \cdots$, the products $E_1(x)^{\nu_1}\cdots E_m(x)^{\nu_m}$ $(\nu_1+\cdots+\nu_m\leqq\nu)$ form a normal system. Let $\alpha\neq 0$ be an algebraic number different from the poles of the functions $Q_{kl}(x)$ $(k, l=1, \cdots, m)$. Then the numbers $E_1(\alpha), \cdots, E_m(\alpha)$ are algebraically independent. As a special case, this theorem contains the author's earlier transcendency theorem about Bessel functions [Abh. Preuss. Akad. Wiss., Phys.-Math. Kl. 1929, 1–70 (1930)], and, in fact, the proof uses the same ideas as this earlier paper.

Chapter 3 contains the two proofs of Schneider and Gelfond of the transcendency of a^b for irrational algebraic b and algebraic $a\neq 0$, $\neq 1$. Of particular interest is the very short form given to Gelfond's proof. In the last chapter, the following theorem of Schneider is proved. Let $\mathfrak{p}_1=(\xi_1, \eta_1)$ and $\mathfrak{p}_2=(\xi_2, \eta_2)$ be two points on the Riemann surface $\mathfrak{R}$ of a curve $\varphi(\xi, \eta)=0$ of genus 1 with algebraic coefficients.

Let ξ_1, η_1 and ξ_2, η_2 be algebraic numbers. Let $w(\mathfrak{p})$ be an indefinite elliptic integral of the second kind which is regular at $\mathfrak{p}_1$ and $\mathfrak{p}_2$ and is not a rational function. Then

$$w(\mathfrak{p}_2)-w(\mathfrak{p}_1)=\int_{\mathfrak{p}_1}^{\mathfrak{p}_2}dw$$

is transcendental unless $\mathfrak{p}_1$ and $\mathfrak{p}_2$ coincide and the path of integration on $\mathfrak{R}$ is homologous to 0. This result contains, e.g., the transcendency of the perimeter of an ellipse with algebraic axes. The booklet ends with a rather short bibliography. *K. Mahler* (Manchester).

Dietrich, Verne E., and Rosenthal, Arthur. Transcendence of factorial series with periodic coefficients. Bull. Amer. Math. Soc. **55**, 954–956 (1949).

The authors obtain the following result from Lindemann's general theorem. If the coefficients a_n in the series $\varphi(z)=\sum_{n=0}^{\infty}a_nz^n/n!$ are algebraic and form a periodic sequence from some a_n on, then $\varphi(z)$ is a transcendental number for every algebraic value of $z\neq 0$, except in the trivial case that $\varphi(z)$ is a polynomial in z. [This theorem is contained in a more general result of B. McMillan, J. Math. Physics **18**, 28–33 (1939); see theorem II, p. 30, and remark 2, p. 33. However, McMillan's proof is not quite correct and his results need supplementing.] *J. Popken* (Utrecht).

Schmidt, Hermann. Zur Approximation und Kettenbruchentwicklung quadratischer Zahlen. Math. Z. **52**, 168–192 (1949).

The author generalizes Heron's method for the computation of the roots of positive integers to complex numbers. He then considers a ring R of complex numbers with certain properties and uses the method mentioned above to obtain approximations for $\sqrt{D}$, D belonging to the ring R, by numbers of the quotient field of R. He then specializes to integers in complex quadratic fields of class number 1. Using methods from the theory of continued fractions, he gives a condition necessary and sufficient in order that a quadratic irrational ξ has an infinity of rational approximations y/x, such that $|\xi-y/x|<x^{-2}/M(\xi)$, $M(\xi)$ denoting Perron's "modular" function [see Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 29].

Next, continued fractions of integral quadratic irrationals with positive different are considered. Among other things a condition is given which is necessary and sufficient for a continued fraction algorithm to represent a number of the class considered. This is an extension of a well-known theorem of Euler and Muir concerning continued fractions of the roots of positive integers. Finally the author derives some general theorems concerning the form of the continued fraction algorithm of quadratic irrationals. For instance, he shows by applying Kronecker's approximation theorem that, if ξ denotes a real quadratic irrational with positive different, then there exist continued fractions for numbers of the form $n\xi$ $(n=1, 2, \cdots)$ with periods of arbitrary large length. *J. Popken* (Utrecht).

Chalk, J. H. H. Reduced binary cubic forms. J. London Math. Soc. **24**, 280–284 (1949).

Let $f(x, y)=ax^3+bx^2y+cxy^2+dy^3$ have distinct real linear factors. Its discriminant $D>0$ and its quadratic covariant $Q(x,y)=Ax^2+Bxy+Cy^2$ has discriminant $-3D$. Assume that f is reduced, i.e., that Q is. Then $A\leqq D$. Davenport [same J. **20**, 14–22 (1945); these Rev. **7**, 418] has shown that $\min\{|f(1, 0)|, |f(0, 1)|, |f(1, 1)|, |f(1, -1)|\}\leqq(D/49)^{\frac{1}{4}}$ with equality only when $f(x, y)$ is a multiple of

$$x^3+x^2y-2xy^2-y^3$$

or $x^3+2x^2y-xy^2-y^3$. The author improves this by replacing D by A^2. He also finds all reduced f for which $A^2=D$. *L. Tornheim* (Ann Arbor, Mich.).

Chalk, J. H. H., and Rogers, C. A. The successive minima of a convex cylinder. J. London Math. Soc. **24**, 284–291 (1949).

Davenport [Nederl. Akad. Wetensch., Proc. **49**, 822–828 = Indagationes Math. **8**, 525–531 (1946); these Rev. **8**, 565] has conjectured that if $\lambda_1, \cdots, \lambda_n$ are the successive minima for a lattice Λ of a convex region S in n-dimensional space and $\Delta(S)$ is the critical determinant of S, then $\lambda_1\cdots\lambda_n\Delta(S)\leqq d(\Lambda)$, the determinant of Λ. This has been proved true when $n=2$, when S is a generalized ellipsoid, or when the volume of S is $2^n\Delta(S)$. Here it is proved when $n=3$ and S is a convex symmetrical cylinder. *L. Tornheim* (Ann Arbor, Mich.).

Schneider, Theodor. Über einen Hlawkaschen Satz aus der Geometrie der Zahlen. Arch. Math. **2**, 81–86 (1950).

Let $k_1, \cdots, k_n$ be positive numbers, M a bounded, closed, Jordan measurable set, and $M(K)$ the set of points $((x_1-y_1)/k_1, \cdots, (x_n-y_n)/k_n)$, where $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are in M. If M has a volume not less than $\frac{1}{2}k_0k_1\cdots k_n$, k_0 being a positive integer, and if $M(K)$ has at most k_0-1 pairs of symmetric lattice points distinct from the origin in its interior, then for every point $r=(r_1, \cdots, r_n)$ there exist at least k_0 lattice points g such that $r+g$ is in $M(K)$. This improves a theorem of Hlawka [Math. Z. **49**, 285–312 (1943); these Rev. **5**, 201]. The proof is essentially arithmetical. A refinement of a theorem of van der Corput [Acta Arith. **2**, 145–146 (1936)] on the number of lattice points in $M(K)$ is made and then applied to the set M' of points $(x_1', \cdots, x_{n+1}')$ where $x_i'=x_i+(r_ik_i/k_{n+1})x_{n+1}$ $(i=1, \cdots, n)$ and $x_{n+1}'=x_{n+1}$ with $(x_1, \cdots, x_n)$ in M and $|x_{n+1}|\leqq k_{n+1}$. *L. Tornheim* (Ann Arbor, Mich.).

Erdös, P., and Koksma, J. F. On the uniform distribution modulo 1 of sequences $(f(n, \theta))$. Nederl. Akad. Wetensch., Proc. **52**, 851–854 = Indagationes Math. **11**, 299–302 (1949).

As a special case of a still more general theorem, the following result is proved. Let $f(n, \theta)$ be defined for $n=1, 2, \cdots$ and all θ with $\alpha\leqq\theta\leqq\beta$; let $(\partial/\partial\theta)f(n, \theta)$ be continuous in θ; and let $(\partial/\partial\theta)f(n_1, \theta)-(\partial/\partial\theta)f(n_2, \theta)$, for $n_1\neq n_2$, be monotonic (increasing or decreasing) in θ and of absolute value at least δ, where $\delta>0$ is independent of n_1, n_2, and θ. Then for $\epsilon>0$ and almost all θ the discrepancy $D(N, \theta)$ of the sequence $f(1, \theta), f(2, \theta), f(3, \theta), \cdots$ satisfies the inequality $ND(N, \theta)=O(N^{\frac{1}{2}}\log^{(5/2)+\epsilon}N)$. *K. Mahler* (Manchester).

Erdös, P. On the converse of Fermat's theorem. Amer. Math. Monthly **56**, 623–624 (1949).

A pseudoprime is a positive integer n, not a prime, such that $2^n\equiv 2 \pmod{n}$. Generalizing a method due to Lehmer, the author proves that for any $k\geqq 2$ there are infinitely many pseudoprimes which are products of exactly k different primes. The proof is constructive, proceeding from $k-1$ to k. Other results are stated, and some problems are propounded. *H. Davenport* (London).

Moser, Leo. A theorem on the distribution of primes. Amer. Math. Monthly 56, 624–625 (1949).

The author gives a short and elementary (though somewhat condensed) proof that for any positive integer r there is a prime p satisfying $3\cdot 2^{2r-1}<p<3\cdot 2^{2r}$, and indicates how the proof can be modified to give Bertrand's postulate. [The condition in (4) should read: $p^a \leqq 2n < p^{a+1}$.]

H. Davenport (London).

Nyman, Bertil. A general prime number theorem. Acta Math. 81, 299–307 (1949).

This is a development of Beurling's generalised prime number theory [Acta Math. 68, 255–291 (1937)]. The "primes" are the members y_n of a given real sequence $(1<y_1<y_2<\cdots; y_n\rightarrow\infty)$, and the "numbers" are all possible products x_n of them $(1<x_1\leqq x_2\leqq\cdots)$ with repetitions for multiple representations. Let $\pi(x)$ be the number of $y_n\leqq x$, $N(x)$ the number of $x_n\leqq x$, and $\zeta(s)=1+x_1^{-s}+x_2^{-s}+\cdots$ $(s=\sigma+it,\ \sigma>1)$. It is asserted that the following three statements are equivalent: (A) $N(x)=ax+O(x(\log x)^{-n})$ as $x\rightarrow\infty$, for some fixed $a>0$ and every fixed $n>0$; (B) $|\zeta^{(n)}(s)|<A|t|^{\epsilon}$, $|\zeta(s)|^{-1}<A|t|^{\epsilon}$ $(\sigma>1,\ |t|\geqq\epsilon)$, for every $\epsilon>0$, every integer $n\geqq 0$, and some $A=A(\epsilon, n)$; (C) $\pi(x)=\mathrm{Li}\,(x)+O(x(\log x)^{-n})$ as $x\rightarrow\infty$, for every fixed $n>0$. The argument is based on the scheme of implications: (1) $A\twoheadrightarrow B$, (2) $B\twoheadrightarrow C$, (3) $C\twoheadrightarrow B$, (4) $B\twoheadrightarrow A$. Of these (1) and (3) are established by partial integration and, in case (1), Hadamard's classical deduction from the inequality $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|\geqq 1$ $(\sigma>1)$; while (2) and (4) are based on Parseval's formula. By the use of derivatives of high order the operations are confined essentially to the open half-plane $\sigma>1$. [The argument does not seem to correspond logically to the enunciation, for under the headings $B\twoheadrightarrow C$ and $B\twoheadrightarrow A$ the author proves only $A\&B\twoheadrightarrow C$ and $C\&B\twoheadrightarrow A$, respectively. Indeed, B can hardly be equivalent by itself to A or C, since it does not specify the behaviour of $\zeta(s)$ near $s=1$. Thus, if $y_n=2^n$, B is true but A and C are false. The main equivalence $A\leftrightarrow C$ is, however, valid.]

A. E. Ingham (Cambridge, England).

Rosser, J. Barkley. Real roots of Dirichlet L-series. Bull. Amer. Math. Soc. 55, 906–913 (1949).

Let χ be a real non-principal character modulo k. It is shewn that, for $2\leqq k\leqq 67$, $L(s,\chi)$ has no positive real zeros. For these values of k, therefore, the error terms in the asymptotic formulae for $\pi(k, l; x)$ can be improved considerably since they depend upon the abscissae of real zeros of such functions $L(s,\chi)$ [see A. Page, Proc. London Math. Soc. (2) 39, 116–141 (1935)]. The author expresses $L(s,\chi)$ as a sum

$$\sum_{M=1}^{\infty} f(s,k,M)\Sigma_M$$

where, for $s>0$, $f(s,k,M)$ is positive and

$$\Sigma_M=\sum_{n=1}^{[\frac{1}{2}k]}\chi(n)(k-2n)^M.$$

When considering the positive zeros of $L(s,\chi)$ attention may be confined to primitive characters. It is stated that for all such characters $\chi(n)$ with $k\leqq 67$, except $k=43$ and $k=67$, it is possible to shew, by grouping terms suitably, that $\Sigma_M\geqq 0$; hence $L(s,\chi)\neq 0$ for $s>0$. See also the work of S. Chowla [Acta Arith. 1, 113–114 (1935)]. For $k=43$ or 67, $\Sigma_3<0$ but it can be shewn that the initial positive terms in the infinite series outweigh the negative terms. This is demonstrated in detail for $k=67$; it is stated that the proof for $k=43$ is similar and easier. The author also states that his method has been tried on all $k\leqq 227$ and that it has been ascertained that, except for the cases $k=148$ and $k=163$, $L(s,\chi)$ has no positive real zeros for $2\leqq k\leqq 227$. The cases $k=148$ and $k=163$ are now being studied.

R. A. Rankin (Cambridge, England).

Whiteman, Albert Leon. Theorems on quadratic partitions. Proc. Nat. Acad. Sci. U. S. A. 36, 60–65 (1950).

Let $f(k)$ and $F(k)$ represent the number of distinct solutions of $x^3+x\equiv k \pmod p$ and $x^5+x^3\equiv k \pmod p$, respectively. Let g be a primitive root of a prime p. In case $p=3f+1$, the unique solution with $a\equiv 1 \pmod 3$ and $b\equiv 0 \pmod 3$ of $4p=a^2+3b^2$ is given by $a=-p+\sum_{s=0}^{p-1}f(s^3)$ and $3b=\sum_{s=0}^{p-1}\{f(gs^3)-f(g^2s^3)\}$. In case $p=7f+1$, the unique solution of $4p=a^2+7b^2$ with $a\equiv 5 \pmod 7$ is given by $a=-p+\sum_{s=0}^{p-1}F(s^7)$ and $7b=\sum_{s=0}^{p-1}\{F(gs^7)-F(g^6s^7)\}$. Problems of this kind were previously treated by the author in terms of Jacobsthal sums [Duke Math. J. 16, 619–626 (1949); these Rev. 11, 230] but those methods seem to be inapplicable to the case $p=7f+1$.

I. Niven.

Verdenius, W. On problems analogous to those of Goldbach and Waring. Nederl. Akad. Wetensch., Proc. 52, 725–733 = Indagationes Math. 11, 255–263 (1949).

This paper is a summary of the author's thesis [Over problemen analoog aan die van Goldbach en Waring, Amsterdam, 1948]. It deals with the representation of m integers $t_1, \cdots, t_m$ in the form

$$t_\mu=\sum_{\nu=1}^{n} b_{\mu\nu}\psi_\nu(y_{\nu 1},\cdots,y_{\nu s_\nu}),\qquad \mu=1,\cdots,m,$$

where $m, n, s_1, \cdots, s_n$ are fixed positive integers, the $b_{\mu\nu}$ are fixed integers, the ψ_ν are fixed integral-valued polynomials, and the $y_{\nu\sigma}$ take on prime values in certain intervals. The theorems stated are too complicated to quote in detail. The author bases his work on van der Corput's version of the Hardy-Littlewood-Vinogradov method [Acta Arith. 3, 180–234 (1939)].

P. T. Bateman (Princeton, N. J.).

Èpel'baum, B. The construction of a basis of G. F. Voronoi's type for a field of algebraic numbers. Doklady Akad. Nauk SSSR (N.S.) 64, 637–640 (1949). (Russian)

For ρ an algebraic integer over the field of rational numbers K, the author considers the construction of a certain kind of minimal basis for the field $K(\rho)$. It is noted that $K(\rho)$ has a minimal basis of the form

$$\omega_i=D_i^{-1}\{b_{i1}+b_{i2}\rho+\cdots+b_{i,i-1}\rho^{i-2}+\rho^{i-1}\},\qquad i=1,\cdots,n;\ \omega_1=1,$$

where the b_{ij} and D_i are rational integers and D_i divides D_{i+1}. The "normal" basis which the author then proceeds to construct is one such that both $\omega_2\omega_i+f_i=(D_{i+1}/D_i)\omega_{i+1}$, $i=3,\cdots,n-1$, and $\omega_2\omega_n=$ rational integer, where the f_i are rational integers. This generalizes an analogous result of Voronoi for cubic fields.

H. N. Shapiro.

***Irminger, Hans. Beiträge zu einem rein arithmetischen Aufbau der Theorie der Klassenkörper.** Thesis, University of Zürich, 1947. 75 pp.

The first part proves the arithmetic inequality of class field theory for cyclic fields of odd prime degree, and includes a detailed study of the norm residue groups modulo all powers of ramified primes. The second part is supposed to prove the existence theorem, for ideal class groups which

are of odd prime order and satisfy certain further conditions. However, this proof is based on an incorrect formula for the Hilbert ramification number. [The field of 9th roots of unity is one of many counterexamples for the author's theorems 25 and 26.] *G. Whaples* (Bloomington, Ind.).

Braun, Hel. Hermitian modular functions. Ann. of Math. (2) **50**, 827–855 (1949).

The Hermitian modular group of degree n is defined as the group of all matrices

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(where A, B, C, D are matrices with n rows and columns whose elements are integral numbers of a given imaginary quadratic field $\mathfrak{K}$ of discriminant d) satisfying $\bar{M}JM=J$, where

$$J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

E is the n-rowed unit matrix and $\bar{M}$ is the transpose conjugate complex to M. The Poincaré-Eisenstein series belonging to this group are defined as follows. A pair of n-rowed square matrices A and B with elements in $\mathfrak{K}$ is termed a Hermitian pair if the matrix (AB) has rank n and if $A\bar{B}=B\bar{A}$. It is called integral if the elements of A and B are integral. It is called coprime if it is integral and if for any row vector x the assumption that xA and xB are integral implies that x is integral. Two coprime Hermitian pairs of A, B and A_1, B_1 of $\mathfrak{K}$ are termed associate (or are said to belong to the same class) if $A_1=UA$, $B_1=UB$, where U is unimodular (a matrix U of $\mathfrak{K}$ is called unimodular if it is integral and has an integral inverse U^{-1}). Now if Z is an n-rowed square matrix whose elements are complex variables and such that $i^{-1}(Z-\bar{Z})$ is a matrix of a positive Hermitian form define $\varphi_g(Z)=\sum_{A,B}|AZ+B|^{-g}$, where A, B run over a complete system of class representatives of coprime n-rowed Hermitian pairs of $\mathfrak{K}$ ($|AZ+B|$ is the determinant of the matrix $AZ+B$). It is supposed that g is an even positive integer and that $4|g$ if $d=-4$, $6|g$ if $d=-3$. Each class of Hermitian pairs of $\mathfrak{K}$ is proved to contain coprime pairs. The author proves that the series of the absolute values of the terms of the series $\varphi_g(Z)$ converges for all real values of $g>2n$, but diverges for $g=2n$. Writing $Z=\dot{Z}+\omega\ddot{Z}$, $Z'=\dot{Z}+\bar{\omega}\ddot{Z}$, where $\omega=\frac{1}{2}(d+\sqrt{d})$, $\bar{\omega}=\frac{1}{2}(d-\sqrt{d})$ and where Z' is the transposed matrix of Z, the matrices $\dot{Z}=(\dot{Z}_{kl})$ and $\ddot{Z}=(\ddot{Z}_{kl})$ have the properties $\ddot{Z}'=-\ddot{Z}$ and $\dot{Z}'=\dot{Z}+d\ddot{Z}$. Considering $\varphi_g(Z)$ as a function of the variables $\ddot{Z}_{kl}$ $(k<l)$ and $\dot{Z}_{kl}$ $(k\leq l)$ it has the period 1 in each of these variables. It therefore has a Fourier expansion which is of the form $\sum_F a(F)\exp[2\pi i\sigma(FZ)]$, where σ is the trace of the matrix FZ and where $F=(f_{kj})$ runs through all Hermitian matrices of $\mathfrak{K}$ for which f_{kk} and $d^{\frac{1}{2}}f_{kj}$ $(j\neq k)$ are integral. The coefficients $a(F)$ are evaluated and the author proves that they are rational numbers. The proofs are modeled on those of the analogous theorems for modular forms of degree n by C. L. Siegel [Math. Ann. **116**, 617–657 (1939); these Rev. **1**, 203]. *H. D. Kloosterman* (Leiden).

Braun, Hel. Hermitian modular functions. II. Genus invariants of Hermitian forms. Ann. of Math. (2) **51**, 92–104 (1950).

[See the preceding review.] This second part contains the proof of an identity which gives a partial fraction series for the analytic genus invariant of a nonnegative integral Hermitian matrix H of $\mathfrak{K}$. It is an analytic equivalent of the author's main theorem on Hermitian forms [Abh. Math. Sem. Hansischen Univ. **14**, 61–150 (1941); these Rev. **3**, 70]. The identity has the form

$$(1)\quad \sum_K \frac{1}{\mathfrak{M}(H)} \sum_{j=1}^{\lambda} \frac{\mathfrak{A}(H_j, K)}{\mathfrak{A}(H_j)} \exp[2\pi i\sigma(KZ)] = \sum_{A,B} \nu(H, A, B)|AZ+B|^{-g}.$$

On the left K runs through all nonnegative integral n-rowed Hermitian matrices K of $\mathfrak{K}$; the H_j are a complete set of representatives of the classes in the genus of H; $\mathfrak{A}(H, K)$ denotes for given nonnegative matrices H and K the number of representations of K by H, i.e., the number of integral matrices C of $\mathfrak{K}$ such that $\bar{C}HC=K$ and $OC=C$. Here O is a so-called right-idem of H, that is to say an integral matrix of $\mathfrak{K}$ with the same rank r as H and such that $HO=H$ ($\mathfrak{A}(H, K)$ does not depend on the choice of O). Moreover, $\mathfrak{A}(H)$ is the number of representations of H by itself and the measure $\mathfrak{M}(H)$ of the genus of H is defined by

$$\mathfrak{M}(H)=\sum_{j=1}^{\lambda} 1/\mathfrak{A}(H_j).$$

On the right-hand side of (1), A, B run through a complete set of n-rowed non-associate, coprime, positive Hermitian pairs of $\mathfrak{K}$. The coefficients $\nu(H, A, B)$ are of absolute value not exceeding 1 and are (except for numerical factors) generalized Gaussian sums. The identity (1) is the analogue for Hermitian forms of the identity given by Siegel for quadratic forms [same Ann. (2) **38**, 212–291 (1937)].

H. D. Kloosterman (Leiden).

ANALYSIS

Dieudonné, Jean. Sur le polygone de Newton. Arch. Math. **2**, 49–55 (1949).

The author discusses in this note how the theory of Puiseux expansions of algebroid functions may be extended to a class of implicit functions of a real variable. Let $f(x, y)=\sum_{i=1}^{n} a_i x^{\alpha_i} y^{\beta_i}[1+\varphi_i(x, y)]=0$, where the $2n$ real numbers (α_i, β_i) are nonnegative, distinct from the pair $(0, 0)$ with $\min \alpha_i=\min \beta_i=0$, and where the coefficients φ_i are nonvanishing real numbers. Furthermore the functions are continuous in some neighborhood of the point $(0, 0)$ and satisfy $\lim_{(x,y)\to(0,0)} \varphi_i(x, y)=0$. These assumptions are later generalized by replacing the powers x^{α_i} by exponential-logarithmic functions of the type considered by Hardy [Orders of Infinity, 2d ed., Cambridge University Press, 1924], which are bounded and nonnegative in a neighborhood of $x=0$. The assumptions on the functions φ_i are made so that certain function-theoretic arguments for the supplementation of the formal Puiseux arguments can be carried out. Thus it is shown that there exist a finite number of nonnegative real numbers r and λ for which $\varphi(x)\sim rx^{\lambda}$, where $\varphi(x)$ is a continuous positive function on an interval $0\leq x\leq b$ with $\varphi(0)=0$ and $f(x, \varphi(x))=0$. Finally results of Hardy [loc. cit.] on the behaviour at infinity of the solutions of linear differential equations $y'=P(x, y)/Q(x, y)$, with polynomials P, Q, are discussed. *O. F. G. Schilling.*

Tôyama, Hiraku. On the inequality of Ingham and Jessen. Proc. Japan Acad. **24**, 10–12 (1948).

Let $A=(a_{\mu\nu})$ be an $m\times n$ matrix with $a_{\mu\nu}\geqq 0$ (and $a_{\mu\nu}$ not all 0). Let $Q=Q(A, r, s)$ be defined by

$$\left(\sum_{\nu=1}^{n}\left(\sum_{\mu=1}^{m}a_{\mu\nu}^{s}\right)^{r/s}\right)^{1/r}=Q\left(\sum_{\mu=1}^{m}\left(\sum_{\nu=1}^{n}a_{\mu\nu}^{r}\right)^{s/r}\right)^{1/s},$$

$$0<r<s<\infty.$$

The title refers to the symmetrical form of Minkowski's inequality expressed by $1\leqq Q$. It is proved here that $Q\leqq\min(m, n)^{1/r-1/s}$, with equality when $a_{\mu\nu}=1$ $(\mu=\nu)$, 0 $(\mu\neq\nu)$. *A. E. Ingham* (Cambridge, England).

Levi, F. W. Ein Reduktionsverfahren für lineare Vektorungleichungen. Arch. Math. **2**, 24–26 (1949).

Let $\lambda_1, \cdots, \lambda_m$ be linear forms in indeterminates $x_1, \cdots, x_r$ with real coefficients and $k_1, \cdots, k_m$ real numbers. If $0\leqq\sum k_i|\lambda_i|$ for arbitrary real values of the x's, it is proved that the same inequality holds when the x's are interpreted as arbitrary vectors in Euclidean space of any number of dimensions. Applications to geometrical problems are mentioned. *J. M. Thomas* (Durham, N. C.).

m3-4

Petersson, Hans. Über Interpolation durch Lösungen linearer Differentialgleichungen. Abh. Math. Sem. Univ. Hamburg **16**, 41–55 (1949).

Let $f(x)$ have derivatives of orders up to and including n on the interval J. (1) Let $L_n(x, y)=\sum_0^{n-1}a_j(x)y^{(j)}+y^{(n)}$, each $a_j(x)$ real, analytic on J. Let $\varphi(x)$ and $h(x)$ satisfy the conditions $L_n(x, \varphi(x))=0$, $L_n(x, h(x))=n!$, $\varphi^{(k)}(\alpha_j)=f^{(k)}(\alpha_j)$, $h^{(k)}(\alpha_j)=0$ $(0\leqq k\leqq p_j-1,\ \alpha_j\epsilon J,\ j=1, 2, \cdots, r,\ \sum_1^r p_j=n)$. Then

$$f(x)=\varphi(x)+(1/n!)L_n(\xi, f(\xi))h(x), \qquad x\epsilon J,$$

for some ξ on the smallest closed interval containing x and the α_j. For $L_n(x, y)=y^{(n)}$ this is a well-known interpolation formula, which in the case of only one point of interpolation α reduces to Taylor's formula. (2) For the case of only one point of interpolation α the following result is proved:

$$f(x)=\sum_0^{n-1}(1/j!)L_j(\alpha, f(\alpha))u_j(x)+(1/n!)L_n(\xi, f(\xi))h_n(x),\ x\epsilon J.$$

Here $L_0(x, y)=y$, $L_{j+1}(x, y)=b_j(x)L_j(x, y)+(d/dx)L_j(x, y)$, $(j=0, 1, 2, \cdots)$, where the $b_j(x)$ are arbitrarily given real analytic functions of x on J. The $u_j(x)$ satisfy $L_j(x, u_j(x))=j!\exp\left(-\int_\alpha^x b_j(t)dt\right)$, $u_j^{(k)}(\alpha)=0$ $(0\leqq k\leqq j-1)$ (hence $L_{j+1}(x, u_j(x))=0$, $u_j^{(j)}(\alpha)=j!$). Finally,

$$L_n(x, h_n(x))=n!,\quad h_n^{(k)}(\alpha)=0\ (0\leqq k\leqq n-1),$$

and $\xi\epsilon[\alpha, x]$. The third formula of the first part of the paper is too complicated to be reproduced here.

In a second part the results under (2) are extended to the complex domain. Let the functions $b_j(x)$, and $f(x)$, be regular for $|x-\alpha|<\rho_0$, while all $|b_j(x)|\leqq M$ on some circle $|x-\alpha|\leqq\rho$, $0<\rho<\min(\rho_0, 1)$. Then

$$f(x)=\sum_0^{\infty}(1/j!)L_j(\alpha, f(\alpha))u_j(x),$$

the series being absolutely and uniformly convergent in every circle $|x-\alpha|\leqq R<\rho/(M(\rho+1))$. No reference is made to the extensive literature on developments $\sum c_ju_j(x)$, $u_j(x)=(x-\alpha)^j(1+\epsilon_j(x))$, of which the preceding one is a special case [see, for example, Boas, Trans. Amer. Math. Soc. **48**, 467–487 (1940); these Rev. **2**, 80; where further references are given]. The author finally remarks that his results may be used to obtain estimates for the difference between the solutions of a nonlinear differential equation and a linear approximating equation. *J. Korevaar.*

Hirschman, I. I., Jr. Proof of a conjecture of I. J. Schoenberg. Proc. Amer. Math. Soc. **1**, 63–65 (1950).

The author proves the following conjecture of I. J. Schoenberg: there does not exist an infinitely differentiable function on the whole axis $f(t)$ vanishing outside an interval, and such that $f^{(n)}(t)$ has exactly n changes of sign. (By Rolle's theorem it follows that if $f(t)\not\equiv 0$ then $f^{(n)}(t)$ has at least n changes of sign). The example $f(t)=\exp[1/(t^2-1)]$ for $|t|<1$ and $f(t)=0$ for $|t|\geqq 1$ (due to D. V. Widder) shows that the theorem is not true if n is replaced by $3n$. *S. Agmon* (Houston, Tex.).

***Korevaar, J. Functions with only monotonic derivatives.** Handelingen van het XXXI^e Nederlands Natuur- en Geneeskundig Congres, pp. 88–89, Haarlem, 1949. (Dutch)

The author gives a simple proof of the theorem that if every derivative of $f(x)$ is monotonic in $(-1, 1)$, then $f(x)$ coincides with a function of $z=x+iy$ which is regular in $|z|<1$. This is a special case of a more difficult theorem of S. Bernstein [Comm. Soc. Math. Kharkow (4) **2**, 1–11 (1928)], whose proof seems to be incomplete. *R. P. Boas, Jr.* (Providence, R. I.).

Agmon, Shmuel. Sur l'équivalence des classes de fonctions indéfiniment dérivables sur un demi-axe. C. R. Acad. Sci. Paris **230**, 350–352 (1950).

Let $\{M_n\}$ and $\{A_n\}$ be two positive sequences, $C\{M_n\}$ and $C\{A_n\}$ the corresponding classes of functions of class C^∞ on a half-line. If $\liminf M_n^{1/n}<\infty$, $C\{M_n\}=C\{1\}$. Suppose then that $\lim M_n^{1/n}=\infty$ and put $N_n=M_nn^n$, $\{N_n^c\}$ the convex regularized sequence of $\{N_n\}$ (with respect to the logarithm), $M_n^d=N_n^c/n^n$. Mandelbrojt has shown that $(M_n^d)^{1/n}=O(A_n^{1/n})$ is a sufficient condition for

$$C\{M_n\}\subset C\{A_n\}.$$

The author shows, by a construction using Laguerre polynomials, that this condition is also necessary. *R. P. Boas, Jr.* (Providence, R. I.).

Bochner, S. Quasi-analytic functions, Laplace operator, positive kernels. Ann. of Math. (2) **51**, 68–91 (1950).

Carleman's theorem on quasi-analytic functions asserts that $f(x)\epsilon C^{(\infty)}[0, \infty]$ is identically zero if (1) $f^{(n)}(0)=0$, $n=0, 1, 2, \cdots$, (2) $\sum_0^\infty[\max|f^{(n)}(x)|]^{-1/n}=\infty$. The present paper is concerned with extensions of such results to other differential operators, especially to the Laplacian and to the Laplace-Beltrami operator. The basic theorem on the circle is: let $f(x+2\pi)=f(x)$, $f(x)\epsilon C^{(\infty)}[0, 2\pi]$; if (1) $f^{(2n)}(x_j)=0$, $j=1, 2$, $x_1-x_2=\alpha 2\pi$, α irrational, $n=0, 1, 2, \cdots$, and if (2) $\sum_1^\infty[\max|f^{(2n)}(x)|]^{-1/n}=\infty$, then $f(x)\equiv 0$. The proof is based on properties of the corresponding boundary value problem for the heat equation, in particular, that the Gauss kernel is positive and that the solution $f(x, t)$ is subject to a maximum principle $\max|f^{(2n)}(x, t)|\leqq\max|f^{(2n)}(x)|$. For these facts the author gives new proofs based upon the observation that the mean value

$$M_{2m}(t)=\left\{(2\pi)^{-1}\int_0^{2\pi}|f(x, t)|^{2m}dx\right\}^{1/2m}$$

is a monotone nonincreasing function of t for every positive integral m. This situation on the circle carries over to an

arbitrary compact space V_k with a positive definite Riemannian metric. The second derivative is then replaced by the Laplace-Beltrami operator $\Delta f = -g^{ki} f_{i,j}$ and the set (x_1, x_2) by a set of determination with respect to Δ, that is, a set U such that if a characteristic function of Δ vanishes on U then it vanishes identically. It is shown that the mean value property holds for the solution $f(x, t)$ of the corresponding heat equation whence it follows that the kernel is positive and the maximum principle holds. Explicit formulas are given for the sphere and for the k-dimensional torus and extensions are indicated to some noncompact situations for the Hermite-Weber, Laguerre and Bessel operators.

Since the basic operators are of the second order while Carleman's theorem really deals with a first order operator, the author considers the question of formulating appropriate theorems for corresponding first order operators which are obtained by extracting square-roots of the Laplacian. These roots may be defined in infinitely many ways, however, and the choice normally affects the corresponding determining set. More generally, the author considers kernels of the form

$$G_\rho(x, y, t) = \sum_{r=0}^{\infty} e^{-\lambda_r^\rho t} \sum \varphi_{rr}(x) \varphi_{rr}(y),$$

where the $\varphi_{sr}(x)$ are characteristic functions corresponding to the characteristic value λ_r of the operator Δ, $0 < \rho \leqq 1$, $0 < t < \infty$, and the operator

$$\Delta^\rho f \sim \sum_{r=0}^{\infty} \lambda_r^\rho \sum \varphi_{rr}(x) \varphi_{rr}(y),$$

which is the negative of the corresponding infinitesimal generator. These kernels are also found to be positive and a theorem of the Carleman type holds for Δ^ρ. A mean value theorem is also noted for solutions of the equation $\Delta f = -\partial^2 f / \partial t^2$; here $M(t)$ is convex and the positivity of the corresponding kernel is related to the Hadamard-Hardy convexity theorem. Finally there are some brief indications of extensions to matrix spaces and related differential operators. *E. Hille* (New Haven, Conn.).

Theory of Sets, Theory of Functions of Real Variables

***Kamke, E. Theory of Sets. Translated by Frederick Bagemihl.** Dover Publications, Inc., New York, N. Y., 1950. vii+144 pp. $2.45.

Translated from the second German edition [de Gruyter, Berlin, 1947; these Rev. **9**, 573].

Tarski, Alfred. Cancellation laws in the arithmetic of cardinals. Fund. Math. **36**, 77–92 (1949).

The following two theorems are proved without the use of the axiom of choice. (I) Given a natural number $m \neq 0$ and two arbitrary cardinals $\mathbf{p}$ and $\mathbf{q}$, if $m\mathbf{p} = m\mathbf{q}$, then $\mathbf{p} = \mathbf{q}$. (II) Given a natural number $m \neq 0$ and two arbitrary cardinals $\mathbf{p}$ and $\mathbf{q}$, if $m\mathbf{p} \leqq m\mathbf{q}$, then $\mathbf{p} \leqq \mathbf{q}$. Using (II), the author gives a purely arithmetical proof of Euclid's theorem for cardinals. If the natural numbers m and n are relatively prime and if $m\mathbf{p} = n\mathbf{q}$, then there is a cardinal number $\mathbf{r}$ such that $\mathbf{p} = n\mathbf{r}$ and $\mathbf{q} = m\mathbf{r}$. These theorems were discovered by A. Lindenbaum, but his proofs were never published and they are now lost. The special cases of (I) and (II) where $m = 2$ were treated by F. Bernstein [Math. Ann. **61**, 117–155 (1905)] and W. Sierpiński [Fund. Math. **3**, 1–6 (1922); **34**, 148–154 (1947); these Rev. **9**, 338]. *B. Jónsson.*

Sierpiński, W. Sur une proposition de A. Lindenbaum équivalente à l'axiome du choix. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 1–3 (1948). (French. Polish summary)

A proof is given of the following theorem which was stated by Lindenbaum, but whose proof was never published [cf. Lindenbaum and Tarski, same C. R. **19**, 299–330 (1930)]. The axiom of choice is equivalent to the following condition P: If $\mathfrak{m}$ and $\mathfrak{n}$ are any two cardinals (not 0) then every class with cardinal $\mathfrak{m}$ can be expressed as the sum of $\mathfrak{n}$ non-empty disjoint classes or every class with cardinal $\mathfrak{n}$ can be expressed as the sum of $\mathfrak{m}$ non-empty disjoint classes. *I. L. Novak* (Wellesley, Mass.).

Sierpiński, W. Sur une famille d'ensembles linéaires singuliers. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 17–21 (1948). (French. Polish summary)

C. Kuratowski [Fund. Math. **8**, 201–208 (1926), in particular, p. 207] proved the existence of a linear (infinite) set E with the following property: if the function f transforms E into a subset in a bicontinuous manner, then $f(x) = x$. The author now proves the following generalization by a method different from that of Kuratowski, but again by means of the well-ordering theorem. There exists a family of 2^c different linear sets E of power c such that if $f(x)$ is a continuous function on E with distinct values and if $f(E) \subset E$, then $f(x) = x$ on E. (Here c designates the power of the continuum.) From this result the author derives another theorem of Kuratowski [loc. cit., p. 203], namely: there exists a family of 2^c linear sets all of which have different types of dimension (in the sense of Fréchet). *A. Rosenthal* (Lafayette, Ind.).

Taĭmanov, A. D. On quasicomponents of disconnected sets. Mat. Sbornik N.S. **25**(67), 367–386 (1949). (Russian)

The author generalizes a theorem of P. S. Novikov [Doklady Akad. Nauk SSSR (N.S.) **56**, 787–790 (1947); these Rev. **9**, 137] in the following way. Let E be any subspace of a Euclidean space R^n. A quasi-component of a point $x \varepsilon E$ is the intersection of all open and closed subsets of E which contain x. [The author misstates this definition.] It is obvious that quasi-components effect a partition of E into disjoint closed subsets. These sets are called 1-quasi-components of E. For every $\alpha < \Omega$, α-quasi-components are defined by induction. If $\alpha = \beta + 1$, and γ-quasi-components $E^\gamma(x)$ have been defined for all $\gamma < \alpha$, then the α-quasi-component of $x \varepsilon E$ is defined as the 1-quasi-component of x in the space $E^\beta(x)$. If α is a limit ordinal, then the α-quasi-component $E^\alpha(x)$ is defined as $\bigcap_{\beta < \alpha} E^\beta(x)$. The decreasing sequence $E^1(x) \supset E^2(x) \supset \cdots \supset E^\alpha(x) \supset \cdots$ $(\alpha < \Omega)$ must be constant from a certain index β onward; and the least such ordinal β is the index of $x \varepsilon E$. It is constant on connected components of E. The main theorem of the paper states that the number of α-quasi-components of any analytic set E in R^n is finite, $\aleph_0$, or $2^{\aleph_0}$. A number of examples of subsets of R^2 with various indices are constructed, and an example due to Mazurkiewicz [Fund. Math. **1**, 61–81 (1920)] is reproduced. Finally, the author shows that every subset of R^2 is an α-quasi-component of some subset of R^2 and discusses conditions for a subset of R^2 to be an α-quasi-component for another subset of R^2. *E. Hewitt.*

Marczewski, Edward. Two-valued measures and prime ideals in fields of sets. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 40 (1947), 11–17 (1948). (English. Polish summary)

The main result of this paper is the extension of S. Ulam's theorem [Fund. Math. 16, 140–150 (1930)] that no non-trivial two-valued measure defined in the field K of all subsets of a one-dimensional interval I is countably additive. The extension consists in replacing K by the field generated by all intervals and one-point sets contained in I. A by-product of the proof is an application of a theorem of Sierpiński concerning the possibility of an effective definition of a nonmeasurable real function.
H. M. Schaerf (Berkeley, Calif.).

Marczewski, E. On the problem of the extension of measures. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 40 (1947), 64–65 (1948). (Polish)

Announcement [without proof] of theorem IV of the author's paper in Colloquium Math. 1, 122–132 (1948) [these Rev. 10, 23]. *H. M. Schaerf* (Berkeley, Calif.).

Kappos, Demetrios A. Die Cartesischen Produkte und die Multiplikation von Massfunktionen in Booleschen Algebren. II. Math. Ann. 121, 223–233 (1949).

Continuing the work of the Carathéodory school and of his own earlier paper [same Ann. 120, 43–74 (1947); these Rev. 9, 178], the author formulates and proves Fubini's theorem for Boolean measure algebras. *P. R. Halmos.*

van Aardenne-Ehrenfest, T. On the impossibility of a just distribution. Nederl. Akad. Wetensch., Proc. 52, 734–739 = Indagationes Math. 11, 264–269 (1949).

Let $A=\{a_n\}$ be a sequence of points in a unit interval I, α a subinterval of I, $|\alpha|$ its length and $A_n(\alpha)$ the number of the $a_m\subset\alpha$ with $m\leqq n$. Let $F_n(A)$ be the upper bound of $|A_n(\alpha)-n|\alpha||$ for α varying in I. The sequence A is said to be "just" if $F_n(A)$ is bounded. The author shows that $F_n(A)$ is never $o(\log\log n/\log\log\log n)$, so that no just distribution exists, and remarks that where $F_n(A)$ has been calculated it has been not $o(\log n)$. *H. D. Ursell.*

Kampé de Fériet, J. Sur un problème d'algèbre abstraite posé par la définition de la moyenne dans la théorie de la turbulence. Ann. Soc. Sci. Bruxelles. Sér. I. 63, 165–180 (1949).

Endeavoring to obtain a more satisfactory definition of mean values in the mechanics of fluids, the author considers the ring R of real functions each of which takes on only a finite (but unbounded) number of values on the domain I considered. A transformation $f\to Tf\epsilon R$ is a "Reynolds transformation" if: $T(f+g)=Tf+Tg$; $T(\lambda 1)=\lambda 1$; $T(fTg)=TfTg$; $f\geqq 0$ implies $Tf\geqq 0$. One and only one such transformation corresponds to each choice of a partition θ_T and scalar set function $\mu(A)$ satisfying

$$\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B),$$

$\mu(0)=0$, $\mu(I)=1$, $0\leqq\mu(A)\leqq 1$; and conversely. In fact T is given explicitly in terms of the coefficients and transforms of the characteristic functions of sets on which f is constant. Incidental result: if π is a partition with only a finite number of classes, then $\pi\cup\theta_T$ is the least partition containing π such that the characteristic function of each class is idempotent and is invariant under T. *P. M. Whitman.*

Kober, H. On a monotone singular function and on the approximation of analytic functions by nearly analytic functions in the complex domain. Trans. Amer. Math. Soc. 67, 433–450 (1949).

The classical example of a monotone continuous singular function $y=\omega(t)$ ($\omega'(t)=0$ almost everywhere) given by Lebesgue, which has been discussed and generalized by Carleman, Hille and Tamarkin, and Gilman, is here further generalized and considered from several new points of view. The intervals of invariance of $\omega(t)$ in $0\leqq t<\infty$ are defined in terms of two parameters $\alpha>\beta\geqq 2$. They constitute for $0\leqq t\leqq 1$ a sequence of sets of intervals such that the mth set consists of $\beta-1$ intervals of length $(\alpha-\beta)(\beta-1)^{-1}\alpha^{-m}$ in each of the gaps left by the previous sets of intervals. The gaps remaining after the mth set of intervals has been designated are each of length α^{-m}. The inverse function is shown to be

$$t=G(y)=\frac{\alpha-\beta}{\beta-1}\sum_{\substack{m>-\infty\\ n\beta^{-m}<y}}^{\infty}\sum_{n=1}^{\infty}\alpha^{-m},$$

n/β not an integer. The function $y=\omega(t)$ satisfies the following five equations which characterize the singular function: $\omega(0)=0$, $\omega(1)=1$, $\omega(t/\alpha)=t/\beta$, $\omega(t)+\omega(1-t)=1$ if $0\leqq t\leqq 1$, $\omega(t+(\alpha-1)/\alpha(\beta-1))=\omega(t)+1/\beta$ if

$$0\leqq t\leqq 1-(\alpha-1)/\alpha(\beta-1).$$

Further,

$$\omega(t+h)\leqq\omega(t)+\omega(h),\quad \omega(t)\leqq t^{\lambda},\quad \omega(t)\geqq((\beta-1)t/(\alpha-1))^{\lambda},$$

where $\lambda=\log\beta/\log\alpha$. It is thus seen that $\omega(t)$ satisfies a Lipschitz condition.

The latter part of the paper contains a study of the approximation of analytic functions of a complex variable by so-called nearly-analytic functions. These functions satisfy a Lipschitz condition in the region considered and have a derivative in the ordinary sense except upon a set of surface measure zero. The following is a nearly analytic function:

$$\gamma(z)=\omega(r)\exp(2\pi i\Omega(\theta/2\pi)),\quad 0\leqq r<\infty,\quad -\infty<\theta<\infty,$$

where $z=re^{i\theta}$. Here $\Omega(t)$ is defined by $\Omega(t)=\omega(t)$, $0\leqq t\leqq 1$, and $\Omega(t+1)-\Omega(t)=1$, $-\infty<t<\infty$. This function is studied in some detail and the derivation of many of its properties given. *E. N. Nilson* (Hartford, Conn.).

Cotlar, M., and Roxin, E. On the variation of discontinuous and multivalued functions of a real variable. Revista Unión Mat. Argentina 14, 38–46 (1949). (Spanish)

Let $f(x)$ be a real, not necessarily single-valued, function of the real variable x. Define $\sup f[E]$ as the least upper bound of $f(x)$ for $x\epsilon E$, and similarly $\inf f[E]$. If $I_\epsilon(x_0)$ is the set $x_0-\epsilon<x<x_0+\epsilon$ let the associated function $\varphi(x)$ to $f(x)$ be determined by the requirement that $y_0=\varphi(x)$ if and only if $\inf f[I_\epsilon(x_0)]\leqq y_0\leqq\sup f[I_\epsilon(x_0)]$ for every $\epsilon>0$. Then $\varphi(x)$ has the Darboux property. For any interval I (closed, open or half open) put $\omega_f(I)=\sup f[I]-\inf f[I]$; for a point x_0 define $\omega_f(x_0)=\limsup f[I_\epsilon(x_0)]-\liminf f[I_\epsilon(x_0)]$, $\epsilon\to 0+$. The total variation $V_f(I)$ of $f(x)$ in an interval I is the least upper bound of $\sum\omega_f(I_\nu)$ for all decompositions of I into disjoint intervals I_ν, where points are admitted among

the I_ν. Then $V_f(I)=V_\varphi(I)$. If $C_f(I)$ is the set of points x_0 in I with $\omega_f(x_0)=0$, then $C_f(I)=C_\varphi(I)$. Finally let $N_f(y_0, M)$ be the set of $x\epsilon M$ for which $y_0=f(x)$. Then $N_f(y_0, C(I))=N_\varphi(y_0, C(I))$ and

$$V_f(I)=\int_{-\infty}^{\infty}N_f(y, C(I))dy+\sum_x\omega_f(x).$$

For single-valued f the set $C(I)$ may be replaced by I.

H. Busemann (Los Angeles, Calif.).

Mossaheb, G. H. On differentiation and Denjoy-behaviour of functions of two real variables. Proc. Cambridge Philos. Soc. **46**, 28–45 (1950).

Let $f(x, y)$ be finite and measurable in a bounded set Q of positive measure, and let E be the set of those points in Q at each of which f is approximately differentiable in a set of directions of positive measure. Then f is approximately differentiable at almost all points of E. However, there are continuous functions which have (ordinary) derivatives in almost all directions in a set of positive measure without being totally differentiable at any point of the set. Denote the upper and lower derivatives of $f(x, y)$ at (x, y) in the direction θ by $\partial^\theta(x, y)$ and $\partial_\theta(x, y)$, respectively. Then θ is called a Denjoy direction for f at (x, y) if $\partial^\theta(x, y)=-\partial^{\theta+\pi}(x, y)$ or $\partial^\theta(x, y)=\partial^{\theta+\pi}(x, y)=\infty$ and $\partial_\theta(x, y)=-\partial_{\theta+\pi}(x, y)$ or $\partial_\theta(x, y)=\partial_\theta(x, y)=-\infty$. A. J. Ward showed that if f is measurable (B) then for almost all (x, y) almost all θ at (x, y) are Denjoy directions. The major part of the present paper is devoted to constructing an example to show that the Denjoy behaviours at the same point may differ: there is a continuous function such that to each point of a set of positive measure there correspond two sets of directions of positive measure in which the Denjoy behaviour of ∂^θ and ∂_θ is different.

H. Busemann (Los Angeles, Calif.).

Haupt, Otto, und Pauc, Christian. Zum Beweise des Verteilungssatzes für Punkte mit unvollständigem Kontingent. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 51–55 (1949).

A descriptive definition of the contingent (generalized tangent) to a set in E_n has been given. A contingent at a point is incomplete if it does not include all of E_n. This paper contains another proof [see F. Roger, Acta Math. **69**, 99–133 (1938); S. Saks, Fund. Math. **26**, 234–240 (1936)] that for an arbitrary set $\mathfrak{M}$ the subset, at each point of which the contingent of $\mathfrak{M}$ is incomplete, is essentially a countable number of pieces each defined by a Lipschitzian function.

J. F. Randolph (Rochester, N. Y.).

Cesari, Lamberto. Sulle superficie di Fréchet. Rivista Mat. Univ. Parma **1**, 19–44 (1950).

The main result is the following principle of local modification for transformations: let T be a transformation defined in a plane domain D and let $C_1 \subset C_2$ be concentric circular subdomains of D; let T_1 be a transformation in C_1 such that (T_1, C_1) is Fréchet equivalent to (T, C_2); then there is a simultaneous extension (T^*, D) of $(T, D-C_2)$ and (T_1, C_1) such that (T^*, D) is Fréchet equivalent to (T_1, D). As an application of this principle, the author gives a new proof of his representation theorem for a closed nondegenerate surface on a circle, and establishes a corresponding representation theorem on the sphere. A further application (to the weak Geöcze problem) is promised.

L. C. Young.

Theory of Functions of Complex Variables

Jabotinsky, Eri. Sur les fonctions inverses. C. R. Acad. Sci. Paris **229**, 508–509 (1949).

With $w=\sum_1^\infty c_n z^n$, $c_1\neq 0$, the integral relations

$$\oint z^m m w^{-n-1}dw+\oint w^m n z^{-n-1}dz=0,$$

$$\oint z^{-n}w^{-1}dw=\oint z^{-n}(w'/w)dz$$

are interpreted as identities between the coefficients of the Laurent series of z^n in powers of w and of w^m (or w'/w) in powers of z. With $z^n=\sum_n^\infty \mu_{n,k}w^k$, these identities are equivalent to $w^{n-1}dw/dz=\sum_n^\infty \mu_{-k,-n}z^{k-1}$. A series of elementary algebraic identities is indicated based on which a direct proof is said to exist.

A. J. Macintyre (Aberdeen).

Valiron, Georges. Sur le rayon de convergence de la série de Lagrange. Bull. Sci. Math. (2) **73**, 116–122 (1949).

This paper is concerned with a result, whose proof was attempted by Rouché in 1862, regarding the Lagrange expansion of the function $y=\varphi(x)$, where x and y are connected by $y=xf(y)$ $(f(0)\neq 0)$. Rouché thought he had proved that on the circumference of the circle of convergence of the Lagrange expansion there must be an algebraic branch-point of $\varphi(x)$. The author shows that, while this is true in many cases, the general statement is not valid. Examples disproving Rouché's statement are constructed.

Z. Nehari (St. Louis, Mo.).

Dehousse, L. Sur les solutions de l'équation $z^n e^{\lambda-z}=1$. Bull. Soc. Roy. Sci. Liége **18**, 213–219 (1949).

The author makes a detailed study of the roots of the indicated equation. His principal conclusion is that when n is odd there are 3, 2 or 1 real roots according as $\lambda>\varphi(n)\equiv n(1-\log n)$, $\lambda=\varphi(n)$, $\lambda<\varphi(n)$; if n is odd, there are 2, 1 or 0 real roots under the same conditions.

R. P. Boas, Jr. (Providence, R. I.).

Andersson, Bengt. On equivalent analytic functions. Ark. Mat. **1**, 77–92 (1949).

Let $\mathfrak{R}$ be the class of functions $w=f(z)$ regular in $|z|\leqq R$. The author defines $f, g\epsilon\mathfrak{R}$ to be equivalent if they can be obtained from each other by reflections or rotations in the z and w planes. For this purpose he defines the M function $\phi_f(r, a)$ of $f(z)$ for $0\leqq r\leqq R$, to be the measure of the set of θ, $0\leqq\theta\leqq 2\pi$, such that $|f(re^{i\theta})|>a$. The author proves (1) that two functions are equivalent if their M functions are equal in an interval $0\leqq r\leqq r_1$ $(r_1>0)$; (2) that if f, g have equal pth means on $|z|=r$ for a sequence $p=p_n$, with $|p_n|\to\infty$, then f, g have equal M functions for this value of r; and hence (3) if f, g have equal M functions for an infinite sequence of distinct $r_i\leqq R$, then f, g are equivalent.

W. K. Hayman (Stanford University, Calif.).

Germansky, Baruch. On the systems of Fekete-points of an arc of a circumference. Riveon Lematematika **3**, 56–57 (1949).

Summary of the author's paper in Hebrew in the same vol., 1–7 (1949); these Rev. **10**, 523.

Eggleston, H. G., and Wilson, R. The coefficient theory of a transcendental singularity of algebraic-logarithmic type. J. London Math. Soc. **24**, 291–304 (1949).

The authors extend theorems by R. Jungen [Comment. Math. Helv. **3**, 266–306 (1931)], R. Wilson [Proc. London

Math. Soc. (2) 42, 196–229 (1936)] and H. G. Eggleston [same vol., 171–181 (1949); these Rev. 11, 168] regarding the Taylor coefficients of functions with algebraic-logarithmic singularities. Studying the special case of the function

$$\sum a_n z^n = (c-z)^{-\sigma}[\log(1-z/c)]^k \sum_{\nu=1}^{\infty} A_\nu (c-z)^{-i\tau_\nu},$$

where σ is a real number, $\log 1=0$, k is a nonnegative integer, $\sum|A_\nu|<\infty$, and the real sequence $\{\tau_\nu\}$ contains one element whose modulus exceeds that of every other element, they obtain the following result: if $\{\eta_n\}$ is a null sequence, there exists a set of indices n, of density one, within which the conditions $|a_n|>|c|^{-n}\eta_n n^{\sigma-1}(\log n)^k$ and $a_n/a_{n+1}=c+o(1)$ are satisfied. This result is used to establish theorems concerning the Taylor coefficients of functions that have several singularities of algebraic-logarithmic type, and concerning the relation between the positions of these singularities and certain determinants similar to the determinants studied by J. Hadamard [J. Math. Pures Appl. (4) 8, 101–186 (1892)]. *G. Piranian* (Ann Arbor, Mich.).

Eggleston, H. G. Notes on Taylor coefficients. II. A further extension of Jungen's theorem. J. London Math. Soc. 25, 58–61 (1950).

[Part I is Eggleston's paper cited in the preceding review.] In the preceding review, the hypothesis "the real sequence $\{\tau_n\}$ contains one element whose modulus exceeds that every other element" may be replaced by the hypothesis "$\{\tau_n\}$ is a bounded real sequence." *G. Piranian.*

Lekkerkerker, C. G. On power series with integral coefficients. I. Nederl. Akad. Wetensch., Proc. 52, 740–746 = Indagationes Math. 11, 270–276 (1949).

The author extends results of A. G. Walker [J. London Math. Soc. 19, 106–107 (1944); these Rev. 6, 263] and of H. Graetzer [ibid. 22, 90–92 (1947); these Rev. 9, 179]. His extension is that if r is a positive integer, if $p_1, p_2, \cdots, p_r$ are different from zero with $-1<p_1<p_2<\cdots<p_r<1$ and if $\eta_1, \eta_2, \cdots, \eta_r$ are arbitrary real numbers, then there exists a power series $f(x)=\sum_0^\infty a_n x^n$ with bounded integral coefficients a_n, such that $f(p_m)=\eta_m$, $m=1, 2, \cdots, r$. In the particular case $r=1$, it is shown also that

$$(1) \qquad |a_n|<\tfrac{1}{2}|p_1|^{-1}+\tfrac{1}{2}, \qquad n=1, 2, 3, \cdots.$$

Furthermore (1) is sharp in the following sense. All numbers which can be represented by a power series $\sum_0^\infty a_n p_1^n$ with integral coefficients a_n so that

$$|a_n|<\tfrac{1}{2}|p_1|^{-1}-\tfrac{1}{2}, \qquad n\geqq 1,$$

form a set of measure zero. [For an extension in another direction the reviewer calls attention to a result of J. Lehner, Bull. Amer. Math. Soc. 55, 1060 (1949).]

M. S. Robertson (New Brunswick, N. J.).

Petrovitch, Michel. Séries de puissances à coefficients ayant une structure arithmétique. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. 5, 57–64 (1939).

Some remarks of a general character.

S. Mandelbrojt (Paris).

Lehto, Olli. On the existence of analytic functions with a finite Dirichlet integral. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 67, 7 pp. (1949).

Let E be a bounded compact point set in the complex plane such that the complement $\bar{E}$ of E is connected. The author gives a necessary and sufficient condition for the existence, in $\bar{E}$, of nonconstant single-valued analytic functions with a finite Dirichlet integral. However, this condition is very much in the nature of a reformulation of the property concerned. It essentially reduces to the statement that there exist functions with a finite Dirichlet integral in $\bar{E}$ if, and only if, the upper limit of the minima of the Dirichlet integrals in all analytically bounded subdomains of E is finite. [Note: the author uses the term "univalent" with the meaning of "single-valued."] *Z. Nehari.*

Tsuji, Masatsugu. On a regular function, whose real part is positive in a unit circle. Proc. Japan Acad. 21 (1945), 321–329 (1949).

New and modified proofs are given for the well-known theorems of C. Carathéodory, O. Toeplitz, E. Fischer and I. Schur which include among others the following. (1) Let $f(z)=\frac{1}{2}a_0+\sum_1^\infty a_n z^n$ (a_0 real) be regular for $|z|<1$. Then $\Re f(z)\geqq 0$ for $|z|<1$ if, and only if, the Hermitian forms $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ ($a_{-\nu}=\bar{a}_\nu$) are nonnegative for $n=0, 1, 2, \cdots$. If $H_n(x)\geqq 0$ for all n, and $H_n(x)$ is positive definite for $n=0, 1, \cdots, k-1$, $H_k(x)$ positive semi-definite, then

$$(\mathrm{A}) \qquad f(z)=\sum_{\nu=1}^{k}\tfrac{1}{2}r_\nu(1+\epsilon_\nu z)/(1-\epsilon_\nu z),$$

$r_\nu>0$, $|\epsilon_\nu|=1$, $\epsilon_i\neq\epsilon_j$ for $i\neq j$, where k is the rank of the infinite Hermitian matrix H:

$$H=\begin{bmatrix} a_0, a_1, a_2, \cdots \\ \bar{a}_1, a_0, a_1, \cdots \\ \bar{a}_2, \bar{a}_1, a_0, \cdots \\ \cdots\cdots \end{bmatrix}.$$

(2) We have $\Re f(z)\geqq 0$ for $|z|<1$ if, and only if, (i) $\delta_n>0$ for all n or (ii) $\delta_n>0$ for $n=0, 1, \cdots, k-1$ and $\delta_n=0$ for $n\geqq k$ for some k in which case $f(z)$ has the form (A). Here

$$\delta_n=\begin{vmatrix} a_0, a_1, a_2, \cdots, a_n \\ \bar{a}_1, a_0, a_1, \cdots, a_{n-1} \\ \cdots \qquad \cdots \\ \bar{a}_n, \bar{a}_{n-1}, \bar{a}_{n-2}, \cdots, a_0 \end{vmatrix}.$$

In the proof use is made of L. Fejér's theorem that a nonnegative trigonometric polynomial in $(0, 2\pi)$ can be expressed in the form $|\sum_{k=0}^n \gamma_k e^{ik\theta}|^2$. Certain inequalities for Hermitian forms due to I. Schur are used along with Herglotz's Stieltjes integral for functions with nonnegative real part. *M. S. Robertson* (New Brunswick, N. J.).

Tumarkin, G. C. On convergent sequences of analytic functions. Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 163–165 (1949). (Russian)

Results are announced without proofs, dealing with extensions of the following well-known theorem of Ostrowski [Acta Litt. Sci. Szeged 1, 80–87 (1923)] and Khintchine [Fund. Math. 4, 72–75 (1923)]. Let $\{\varphi_n(z)\}$ be a sequence of functions regular in $|z|<1$ and uniformly bounded in their totality there. If the boundary values (in the sense of Fatou's theorem) of the functions of this sequence converge on a set of points of positive measure, then the sequence $\{\varphi_n(z)\}$ converges uniformly in every closed subregion wholly contained in $|z|<1$. Analogous theorems are obtained for meromorphic functions of bounded type, i.e., functions which may be represented as quotients of two bounded regular functions. A connection is also given between the boundary values of the limit function and the limit of the boundary values of the functions of the sequence. In another theorem circular regions are replaced by regions

bounded by rectifiable curves. The paper is marred by a very large number of misprints. *W. Seidel.*

✶Kibbey, Donald Eugene. Boundary Values of Analytic Functions. Abstract of a Thesis, University of Illinois, 1941. ii+9 pp.

Let A be an arc of $|z|=1$, having length ρ, and suppose that for $f(z)$ regular in $|z|<1$, $\liminf |f(z_n)|\geqq 1$ as $n\to\infty$, whenever z_n converges to a point of A. Then $f(z)$ is said to have the property $K(\rho)$. J. L. Doob showed [Ann. of Math. (2) **36**, 117–126 (1935)] that in this case the range of $f(z)$ includes some circle of radius at least $k(\rho)$. The author obtains lower bounds for $k(\rho, M)$, when in addition $|f(z)|<M$. *W. K. Hayman.*

Ghika, Al. Sur une propriété des espaces de fonctions p-sommables ($p>1$). Bull. Math. Soc. Roumaine Sci. **48**, 77–87 (1947).

Let $f(z)$ belong to L_p ($p>1$) on $|z|=1$, and let $\sum_{n=-\infty}^{\infty}a_nz^n$ be the Fourier series of $f(z)$. Let $C_p(\bar{D})$ be the subclass of L_p for which $a_n=0$, $n<0$. The author proves that any $f(z)$ in L_p decomposes uniquely, $f=f_1(z)+\overline{zf_2(z)}$, where $f_1(z)$, $f_2(z)$ belong to $C_p(\bar{D})$. The reviewer remarks that since the conjugate Fourier series $i\{\sum_{n=1}^{\infty}a_{-n}z^{-n}-\sum_{n=1}^{\infty}a_nz^n\}$ belongs to $g(z)$ in L_p, we have at once the required decomposition with $f_1=\frac{1}{2}[a_0+f+ig]$. *W. K. Hayman.*

Goluzin, G. M. On mean values. Mat. Sbornik N.S. **25**(67), 307–314 (1949). (Russian)

The author gives an extension of well-known theorems on weighted means [see, e.g., Hardy, Littlewood, Pólya, Inequalities, Cambridge University Press, 1934, pp. 26, 27]. Let real numbers $p_{k,l}$ and $x_{k,l}$ ($k=1,2,\cdots,n$; $l=1,2,\cdots,m_k$) be such that, for every k, $0<x_{k,1}\leqq x_{k,2}\leqq\cdots\leqq x_{k,m_k}$; $\sigma_{k,s}=\sum_{l=s}^{m_k}p_{k,l}\geqq 0$ ($s=1,2,\cdots,m_k$); and $S=\sum_{k=1}^{n}\sum_{l=1}^{m_k}p_{k,l}>0$. Then, with $S_\lambda=\sum_{k=1}^{n}\sum_{l=1}^{m_k}p_{k,l}x_{k,l}^\lambda$ for $0<\lambda<\infty$, the quantity $m(\lambda)=(S_\lambda/S)^{1/\lambda}$ is a nondecreasing function of λ, while $\log S_\lambda$ is a convex function of λ. On the basis of this theorem the following result is proved. If a finitely connected Riemann surface R which is homeomorphic to a schlicht region lies over the $w=Re^{i\Phi}$ plane, covers the point $w=0$, but does not cover the point $w=\infty$, and is bounded by a finite number of analytic curves, then the quantity

$$M(\lambda)=\left(\int_L R^\lambda d\Phi\Big/\int_L d\Phi\right)^{1/\lambda},$$

where the integrals are taken in the positive sense over the complete boundary L of R, is a nondecreasing function of λ for $0<\lambda<\infty$, while $N(\lambda)=\log\int_L R^\lambda d\Phi$ is convex. The proof depends on approximating the above integrals by corresponding Riemann sums. This theorem is then applied to give the following result. Let $w=f(z)$, $f(0)=0$ be regular and p-valent in $|z|<1$ and let $M_\lambda(r)=(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^\lambda d\theta$, $\lambda>0$. If for some $\lambda>0$ and all r in $0<r<1$ the inequality $M_\lambda(r)\leqq m(r)$ holds, where $m(r)$ is continuous together with its first derivative $m'(r)$ for $0\leqq r<1$ and $m(0)=0$, $m'(r)$ nondecreasing, then for $0\leqq\lambda'<\lambda$ and $0<r<1$

$$M_{\lambda'}(r)\leqq p\lambda'(p\lambda)^{-\lambda'/\lambda}\int_0^r[m'(\rho)]^{\lambda'/\lambda}\rho^{-1+\lambda'/\lambda}d\rho.$$

Specialization of this result yields an inequality of Biernacki [C. R. Acad. Sci. Paris **203**, 449–451 (1936)]. Two slight errors do not invalidate the proofs. *W. Seidel.*

Lohin, I. F. Remarks on estimates for regular functions. Mat. Sbornik N.S. **24**(66), 249–262 (1949). (Russian)

Let S be the class of functions $F(z)$ which are regular and schlicht in $|z|<1$, with $F(0)=0$, $F'(0)=1$. The following three classes of functions $f(z)$ regular in $|z|<1$ are considered: (1) the class S_L consisting of all functions $f(z)$ subordinate to some $F(z)$ in S; (2) the class K_L consisting of all functions $f(z)$ subordinate to some $F(z)$ in S which maps $|z|<1$ on a convex region; (3) the class C_L of all functions $f(z)$ for which $f(z_1)\cdot f(z_2)\neq 1$ for arbitrary z_1, z_2, in $|z|<1$. Using results and methods of Golusin [Rec. Math. [Mat. Sbornik] N.S. **16**(58), 295–306 (1945); these Rev. **7**, 202] and Rogosinski [J. London Math. Soc. **14**, 4–11 (1939)], the author obtains various estimates for the derivatives and coefficients of functions of the above three classes. For example, if $f(z)\in S_L$ and has the expansion $f(z)=a_mz^m+\cdots$, $m\geqq 1$, then

$$|f'(z)|\leqq(1+r^m)(1-r^m)^{-3}mr^{m-1},\qquad r=|z|,$$

with equality only for $f(z)=z^m/(1-e^{i\alpha}z^m)^2$. Also, under the above hypotheses $|a_k|\leqq 1$, $k=m, m+1,\cdots,2m-1$, and $|a_{2m}|\leqq 2$. Similar results are obtained for the classes K_L and C_L. Some subclasses of C_L are also investigated.
W. Seidel (Rochester, N. Y.).

Milin, I. M., and Lebedev, N. A. On the coefficients of certain classes of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **67**, 221–223 (1949). (Russian)

A number of results are announced without proof. Let $f(z)=z+\sum_{n=2}^{\infty}c_nz^n$ be regular and schlicht in $|z|<1$. Then the following inequalities hold:

$$(2\pi)^{-1}\int_{|z|=1}|f(rz)|\,|dz|<r/(1-r^2)+0.65,\qquad 0<r<1;$$

$$|c_n|<\tfrac{1}{2}en+1.8,\qquad n=2,3,\cdots.$$

The latter is in many cases an improvement over recent inequalities of Bazilevič [same Doklady (N.S.) **65**, 253–255 (1949); these Rev. **10**, 602]. An inequality for functions of the above class given by Gronwall [Proc. Nat. Acad. Sci. U. S. A. **6**, 300–302 (1920)] is shown to be incorrect and is replaced by

$$|f(z)|\leqq r(1-r)^{-2}\exp\{-\tfrac{1}{2}(2-|c_2|)r^2\},$$
$$|f'(z)|\leqq(1+r)(1-r)^{-3}\exp\{-\tfrac{1}{2}(2-|c_2|)r^2\},$$

where $r=|z|$ and $0<r<1$. It is also stated that if $f(z)=\sum_{n=1}^{\infty}a_nz^n$ is regular in $|z|<1$ and such that $f(z_1)\cdot f(z_2)\neq 1$ for arbitrary z_1, z_2 in $|z|<1$, then $|a_n|\leqq 1$, $n=1,2,\cdots$ with the sign of equality holding only for $f(z)=\eta z^n$, $|\eta|=1$. This is a conjecture of Rogosinski [J. London Math. Soc. **14**, 4–11 (1939)] who proved it only for $n=1, 2$. Further results concerning functions of related classes are also announced. *W. Seidel.*

Capelli, Pedro. Some observations on univalent functions and a particular class of them. Fac. Ci. Mat. Univ. Nac. Litoral. Publ. Inst. Mat. **8**, 195–223 (1948). (Spanish)

Let U be the class of functions $f(z)=z+a_2z^2+\cdots$ which are univalent in $|z|<1$, and let $M(z)=z/(1-z)^2$. Denote by $A(r,f)$ the area of the image of $|z|\leqq r<1$ under $f(z)$. The author conjectures that for $f(z)\in U$, (1) $A(r,f)\leqq A(r,M)$ and (2) $\limsup_{r\to 1}A(r,f)/A(r,M)\leqq 1$. These follow from the conjectures (3) $|a_n|\leqq n$ and (4) $\limsup|a_n|/n\leqq 1$, respectively. The effect of (1) and (2) on (3) and (4) is discussed.

Let L_1 be the class of functions $f(z)\epsilon U$, having for $|z|=1$ a single infinity at $z=1$. Let L^* be the class of functions $f(z)\epsilon U$, mapping the boundary $|z|=1$ into a curve contained in the half plane $\Re(w)<0$. Let $L_1^*=L_1\cap L^*$. The author proves the following. (I) For each $f(z)\epsilon U$, there are a γ and a β such that $\gamma f(z)/(\gamma-f(z))\epsilon L_1$ and $\beta f(z)/(\beta-f(z))\epsilon L^*$. (II) If $f(z)\epsilon L_1$ and $(1-z)^2f(z)$ is uniformly continuous for $|z|<1$, then (4) is satisfied. (III) If $f(z)\epsilon L_1^*$ and $f_1(z)=\{f(z^2)\}^{\frac{1}{2}}=z+b_3z^3+\cdots$ then $|b_n-b_{n-2}|\leqq 2$ and $\lim (b_n-b_{n-2})=0$. (IV) If $f(z)\epsilon L_1^*$ then $|a_{n+k}-2a_n+a_{n-k}|\leqq 2$ for $n=k, k+1, \cdots$ and $\lim (a_{n+k}-2a_n+a_{n-k})=0$.

If in defining the classes L_1 and L^*, the single infinity may be anywhere in $|z|=1$, or the image of $|z|=1$ is required merely to lie in some half plane [and these are the author's definitions] then (IV) and the second part of (III) are false, as the example $z/(1+z)^2$ shows.

A. W. Goodman (Lexington, Ky.).

Kobori, Akira. An evaluation in the theory of multivalent functions. Jap. J. Math. 19, 275–285 (1948).

Let $w(z)=z^p+a_{p+1}z^{p+1}+\cdots$ be regular and p-valent in $|z|<1$. Then $w(re^{i\theta})\geqq r^p/k^p(1+r)^{2p}$, where $k=1.00755\cdots$ for $0\leqq r\leqq .80458\cdots$ and $k=1.03142$ for $.80458\cdots<r\leqq 1$.

A. W. Goodman (Lexington, Ky.).

See the note to p. 241, Tsuji

Macintyre, Sheila Scott. On the zeros of successive derivatives of integral functions. Trans. Amer. Math. Soc. 67, 241–251 (1949).

Whittaker [Interpolatory Function Theory, Cambridge University Press, 1935] proved that there is a number W such that if f is of exponential type less than W, and $f^{(n)}(z_n)=0$ for a sequence of points $\{z_n\}$ with $|z_n|\leqq 1$, then $f\equiv 0$. Let $G_n(z)=G_n(z; z_1, z_2, \cdots, z_n)$ be the Gontcharov polynomials. Levinson [Duke Math. J. 11, 729–733 (1944); these Rev. 6, 122] showed that $\max |G_n(z_0)|\leqq \gamma^{n+1}$, taken over all z_n with $|z_n|\leqq 1$, and that $1/\gamma\leqq W$; by computation, he obtained the estimate $\gamma<1.389$. By an improved procedure, the author has obtained the estimate $\gamma<1.3775$; thus, $.7259<W<.7378$, which still permits the conjecture $W=2/e$. The corresponding problem with $\{z_n\}$ real was solved by I. J. Schoenberg [same Trans. 40, 12–23 (1936)], who showed that the critical type was $\pi/4$. This is proved anew in the present paper, by means of the estimate $\max |G_n(z_0)|\leqq 2(4/\pi)^{n-1}$, where $-1\leqq z_n\leqq 1$. If the points $z_n=x_n+iy_n$ obey $|y_n|\leqq h$ (small), then $(\pi/4)\exp(-\pi^2h/8)$ is a lower bound for the corresponding critical type. The author also observes that since it is only the behavior of z_n for large n which is important, these results also hold if it is merely required that all limit points of $\{z_n\}$ lie in the appropriate set.

R. C. Buck (Madison, Wis.).

Mursi, Zaki. Sur l'ordre de fonctions entières définies par interpolation. Bull. Sci. Math. (2) 73, 96–112 (1949).

The author constructs an entire function $\Phi(z)$ satisfying $\Phi^{(k)}(c_n)=k!a_{nk}$ ($n=0, 1, 2, \cdots$; $k=0, 1, \cdots, \lambda_n$), where c_n, a_{nk} are given complex numbers, $|c_n|\to\infty$, and the λ_n given nonnegative integers. The function $\Phi(z)$ is obtained as a product $G(z)H(z)$, where $G(z)$ is a canonical product formed with the points c_n as zeros of order λ_n+1, and $H(z)$ a meromorphic function with suitably chosen principal parts at the points c_n. Applying results of M. Mursi and C. E. Winn [Quart. J. Math., Oxford Ser. 4, 173–178 (1933)], A. J. MacIntyre and R. Wilson [ibid. 5, 211–220 (1934)] and J. M. Whittaker [Proc. London Math. Soc. (2) 40, 255–272 (1935)] on the order of interpolated entire or meromorphic functions the author obtains an upper bound for the order of $\Phi(z)$, which among other things depends in a complicated way on the behavior of $G(z)$ near the points c_n. [There are many misprints: e.g., the first C_n in theorem 1 should be c_n, C_n and c_r in theorem 2 should be c_n.]

J. Korevaar.

***Plancherel, M. Intégrales de Fourier et fonctions entières.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 31–43. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

This is an exposition of the results proved by the author and G. Pólya [Comment. Math. Helv. 9, 224–248 (1937); 10, 110–163 (1937)]. Unlike the original papers it gives the results in their most general form only at the end.

J. Korevaar (Lafayette, Ind.).

Garabedian, P. R. A problem of Robinson. Bull. Amer. Math. Soc. 55, 917–922 (1949).

Etude du problème suivant: D étant un domaine plan borné dont la frontière C est formée de n courbes fermées analytiques et disjointes, z_0 un point intérieur à D, on désigne par $\Omega(z_0)$ la famille des fonctions $F(z)=\alpha_F/(z-z_0)+h(z)$ holomorphes dans D sauf au pole simple z_0, et satisfaisant à $\lim\sup_{z\to C}|F(z)|\leqq 1$. On pose $\sigma(z)=\max |F(z)|$, $F\epsilon\Omega$, et on étudie l'ensemble $A=E[\sigma(z)=1]$. Si $n=1$, A est vide. Si $n=2$, Robinson [Duke Math. J. 10, 341–354 (1943); ces Rev. 4, 241] a résolu le cas de l'anneau $r<|z|<1$ en montrant que A est le segment $-1<z<-r$. Le cas $n\geqq 3$ est étudié et l'auteur montre que (1) $D-A$ est simplement connexe; (ii) A a en général des points intérieurs. Par contre, si l'on autorise F à avoir $m\geqq n$ poles simples, ou les singularités équivalentes, l'ensemble A est vide.

J. Lelong (Lille).

Garabedian, P. R. Schwarz's lemma and the Szegö kernel function. Trans. Amer. Math. Soc. 67, 1–35 (1949).

L'auteur part du problème suivant: étant donné un domaine fini D du plan z dont la frontière C est formée de n courbes analytiques C_ν, on désigne par $\mathfrak{F}_{z_1}$ la classe des fonctions $f(z)$ régulières dans D sauf au point z_1, et continues sur C, de la forme

$$f(z)=(z-z_1)^{-2}+a_{-1}(z-z_1)^{-1}+a_0+a_1(z-z_1)+\cdots$$

et par Ω_{z_1} celle des fonctions $F(z)$ analytiques dans D, satisfaisant à $|F(z)|\leqq 1$ et de la forme $F(z)=\alpha_F(z-z_1)+\cdots$. On pose $l=\inf l_f$, $\alpha=\sup \alpha_F$ où $l_f=\int_C|f(z)|ds$. Les fonctions extrémales $f_0(z)$, $F_0(z)$ satisfont sur C à la relation $i^{-1}F_0(z)f_0(z)dz(s)\geqq 0$ d'où l'on déduit $\alpha l=2\pi$; les zéros z_i de F_0 résolvent le problème de Jacobi pour D.

Si nous posons $K(z, z_1)=F_0f_0^{\frac{1}{2}}/(2\pi)$ toute fonction $\psi(z)$ régulière dans D et continue sur C satisfait à

$$\psi(z_1)=\int_C\psi(z)\overline{K(z, z_1)}ds$$

et parmi toutes les fonctions $\psi(z)$ satisfaisant à $\psi(z_1)=l$, la fonction $K(z, z_1)/K(z_1, z_1)$ réalise le minimum L de l'intégrale $\int|\psi(z)|^2ds$. Ce noyau $K(z, z_1)$ généralise pour les domaines multiplement connexes le noyau défini par Szegö pour les domaines simplement connexes; et son carré s'exprime au moyen des dérivées $\partial^2G(z, z_i)/\partial z\partial\bar{z}_i$ et des mesures harmoniques des contours C_ν par une formule qui généralise celle de Bergman. Le noyau $K(z, z_1)$ peut être décomposé en série convergente de fonctions ψ_k formant un système orthogonal complet.

Après des applications à un problème de Painlevé (condition d'existence d'une fonction analytique bornée à l'ex-

térieur d'un compact E), l'auteur étudie le problème d'interpolation de Pick-Nevanlinna (détermination des fonctions $F(z)$ satisfaisant à $|F(z)| \leqq 1$ et prenant des valeurs données en k points z_i; étude de l'ensemble Δ des valeurs $F(z_0)$ en un $(k+1)$ième point z_0). *J. Lelong* (Lille).

Jenkins, J. A. Some problems in conformal mapping. Trans. Amer. Math. Soc. **67**, 327–350 (1949).

The author treats conformal mapping problems by means of the method of extremal length of Ahlfors and Beurling [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 341–351 (1947); these Rev. **9**, 23]. Let D be a hexagon 123456, let ρ be a metric in D such that the lengths of curves joining 12, 34; 34, 56; 56, 12 are at least a_1, a_2, a_3. Let $M(a_1, a_2, a_3)$ be the minimum of the corresponding metric area of D for varying ρ. The problem is conformally invariant and canonical forms are obtained for D so that $\rho \equiv 1$. A simpler problem for the pentagon is also treated. It is shown that another hexagon $1'2'3'4'5'6'$ can be mapped into 123456 so that the sides $1'2'$, $3'4'$, $5'6'$ map onto 12, 34, 56 if and only if $M(a_1, a_2, a_3) \geqq M'(a_1, a_2, a_3)$ for all a_1, a_2, a_3. A similar result is deduced for the mapping of one triply-connected domain into another so that the topological situations of the contours correspond. *W. K. Hayman.*

Komatu, Yûsaku. Ein alternierendes Approximationsverfahren für konforme Abbildung von einem Ringgebiete auf einen Kreisring. Proc. Japan Acad. **21** (1945), 146–155 (1949).

Let D be a schlicht doubly-connected domain bounded by $|z|=1$ and a closed curve inside $|z|<1$. The author shows that the conformal mapping of D onto a circular ring can be obtained as the limit of a sequence of mappings, each of which arises from the preceding one by the following two steps. First, the outside of the inner boundary is mapped onto the outside of a circle of suitable radius about the origin; then the inside of the image of the outer boundary obtained by this mapping is mapped onto the inside of the unit circle. *Z. Nehari* (St. Louis, Mo.).

Opitz, Günter. Die Konvergenz des Verfahrens von Theodorsen zur konformen Abbildung kreisähnlicher Gebiete. Arch. Math. **2**, 110–116 (1950).

The problem of finding the function which maps the interior of a simply connected region, which is star-shaped with respect to the origin, conformally on the interior of a unit circle, was reduced by Theodorsen to the problem of solving the Theodorsen [Theodorsen-Garrick] integral equation for a function of one variable,

$$\epsilon(\varphi) = L[\epsilon(\varphi)] = (2\pi)^{-1}\int_0^{2\pi} \log \rho[\alpha+\epsilon(\alpha)] \cot \tfrac{1}{2}(\varphi-\alpha)d\alpha,$$

where $\rho=\rho(\psi)$ is the boundary of the given region in polar coordinates. The Theodorsen method consists of approximating the solution of this equation by iteration, i.e., by calculating the sequence $\epsilon_0(\varphi)=0$, $\epsilon_{n+1}(\varphi)=L[\epsilon_n(\varphi)]$. In this paper the author proves that, if $|\rho'(\psi)/\rho(\psi)|<1$, the sequence $\{\epsilon_n(\varphi)\}_{n=0}^{\infty}$ converges to the solution of the integral equation in the mean, and also proves that this sequence is equicontinuous by proving that the set of numbers $\int_0^{2\pi}[\epsilon_k'(\varphi)]^2d\phi$ $(k=1, 2, \cdots)$ is bounded. From this the author deduces the uniform convergence of the sequence of functions. [These results are essentially a duplication of part of a paper by S. E. Warschawski, Quart. Appl. Math. **3**, 12–28 (1945); these Rev. **6**, 207, apparently not available to the author.] *C. Saltzer* (Cleveland, Ohio).

Lammel, Ernst. Über das Verfahren von Theodorsen zur numerischen Berechnung der Abbildungsfunktion eines einfach zusammenhängenden Bereiches. Monatsh. Math. **53**, 257–267 (1949).

The notation is that of the preceding review. The author's chief result is that the sequence of functions $\{\epsilon_n(\varphi)\}$ converges uniformly to $\epsilon(\varphi)$ under the following hypotheses: (a) $\rho(\theta)$ can be differentiated arbitrarily many times,

(b) $$A = \max_{0\leq\theta\leq 2\pi} |(d/d\theta) \log \rho(\theta)| < 1,$$

and

$$\max_{0\leq\theta\leq 2\pi} |(d^2/d\theta^2) \log \rho(\theta)| \leqq (4\pi)^{-1}(1-A)(1-A^2)^{\frac{1}{2}}.$$

[Warschawski, in the paper cited in the preceding review, proved this theorem under the hypotheses that $\rho(\theta)$ is absolutely continuous, $A<1$, and there is an $a>0$ such that $a/(1+\epsilon) \leqq \rho(\theta) \leqq a(1+\epsilon)$, and in addition, gave an estimate for the accuracy of the approximation of $\epsilon(\varphi)$ by $\epsilon_n(\varphi)$. The paper under review was originally submitted in 1943.] *C. Saltzer* (Cleveland, Ohio).

Ringleb, Friedrich. Numerical and graphical method of conformal representation. National Research Council of Canada, Division of Mechanical Engineering, Technical Translation no. **TT-70**, ii+24 pp. (5 plates) (1948).

[Translated from Zentrale für Wissenschaftliches Berichtswesen des Generalluftzeugmeisters (ZWB), Forsch.-Ber. 1964 (1944).] The first part of this paper is concerned with a modification of the Koebe's "Schmiegungsverfahren" for constructing the conformal mapping function which maps a simply connected region on the interior of a circle. The author proposes replacing Koebe's mapping function whenever possible by a function determined in the following way. Assume that the given region R is contained in a circle and that the center of the circle is contained in R. Let a circular arc be drawn which separates the interior of the circle into two regions, a convex region and a sickle-shaped region, such that R is contained in the latter. The mapping function to be used is the function which maps the interior of the sickle on the interior of the given circle subject to a certain normalization. Estimates for the accuracy of the approximations of the mapping functions calculated by this method are not given although the convergence of the method is proved. The translation is often misleading, e.g., the term "inversion functions" is used when "inverse functions" is meant. The second part of the paper deals with the properties and the construction of small scale conformal representations. *C. Saltzer* (Cleveland, Ohio).

Darwin, Charles. Some conformal transformations involving elliptic functions. Philos. Mag. (7) **41**, 1–11 (1950).

The author gives (mostly without proof) explicit formulas for analytic functions yielding conformal maps of a rectangle onto a variety of domains which may be of interest in applications. All of these formulas involve Jacobian elliptic functions. *Z. Nehari* (St. Louis, Mo.).

Nevanlinna, Rolf. Sur l'existence de certaines classes de différentielles analytiques. C. R. Acad. Sci. Paris **228**, 2002–2004 (1949).

Let $\varphi(z)dz$ $(z=x+iy$ a local parameter) be a linear differential on a Riemann surface F, and write $\Phi=\int\varphi dz$, $D(\varphi)=\iint_F|\varphi|^2dxdy$. The author considers the following four classes of differentials: (1) the class E_1 for which $D(\varphi)<\infty$ and $\Re(\Phi)$ is single-valued; (2) the class E_2 for

which $D(\varphi)<\infty$ and Φ is single-valued; (3) the class E_3 for which $\Re(\Phi)$ is bounded and single-valued; (4) the class E_4 for which Φ is bounded and single-valued. Obviously $E_1\supset E_2$ and $E_3\supset E_4$. The ideal boundary of F is said to have E-measure zero if the class E of differentials contains only the element $\varphi\equiv 0$. Let C_ν $(\nu=1, 2, 3, 4)$ be the class of surfaces F whose boundaries have E_ν-measure zero. Then $C_1\subset C_2$, $C_3\subset C_4$. If P is the class of surfaces F whose boundaries have harmonic measure zero, it is known that a surface of P belongs to C_1 and C_3, hence belongs to $C_1\cap C_2\cap C_3\cap C_4$. If the genus is finite, $P=C_1=C_3$.

Now suppose that F is of hyperbolic type, in which case its boundary has positive harmonic measure. Then F has a Green's function $g(p, q)$ and the level lines $g=\rho>0$ are composed of a finite number of closed curves Γ_ρ^i. Let $h^i(\rho)$ be the harmonic measure of Γ_ρ^i at the point q with respect to the domain $g\geqq\rho$, and let $h(\rho)=\max h^i(\rho)$. Setting $\alpha(\rho)=\int_\rho\{h(t)\}^{-1}dt$, it is stated that $F\subset C_2\cap C_4$ if the integral $\int_0 \exp\{4\alpha(\rho)\}d\rho$ diverges. *D. C. Spencer.*

Pfluger, Albert. Über das Anwachsen eindeutiger analytischer Funktionen auf offenen Riemann'schen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. **64**, 18 pp. (1949).

The author discusses the growth of the Dirichlet integral and maximum modulus of analytic functions on an open Riemann surface. The paper is a detailed presentation, with some additions, of a previous note on these questions [C. R. Acad. Sci. Paris **229**, 505–507 (1949); these Rev. **11**, 93].
P. R. Garabedian (Stanford University, Calif.).

Pfluger, Albert. Sur l'existence de fonctions non constantes, analytiques, uniformes et bornées sur une surface de Riemann ouverte. C. R. Acad. Sci. Paris **230**, 166–168 (1950).

Continuing earlier work [same C. R. **229**, 505–507 (1949); these Rev. **11**, 93; cf. also the preceding review and L. Sario, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); these Rev. **10**, 365] the author derives the following condition that every bounded analytic function on a Riemann surface F is a constant. Let there be given a conformal metric ds on F, let D_ρ be the set of points with distance less than ρ from a fixed point in F, and suppose that the D_ρ exhaust F and are bounded by $n(\rho)$ Jordan curves $\gamma_k(\rho)$. Let $\Lambda(\rho)$ denote the length of the longest curve $\gamma_k(\rho)$, and set $N(\rho)=\max_{\rho'\leq\rho} n(\rho')$. Then if

$$\limsup_{R\to\infty}\left[4\pi\int_0^R\{\Lambda(\rho)\}^{-1}d\rho-\log N(R)\right]=\infty,$$

each bounded analytic function on F is constant. The proof uses the hyperbolic metric in the unit circle, and is based on the application of Schwarz's inequality to compare length and area. *P. R. Garabedian.*

***Gottschalk, W. H. Conformal Mapping of Abstract Riemann Surfaces.** Published by the author, University of Pennsylvania, Philadelphia, Pa., 1949. v+77 pp. $1.00.

The intent and nature of this discussion are made clear from the following excerpts from the preface. The task of this paper is to prove that every quasi-simple Riemann surface can be conformally mapped onto a slit region. The essential steps of its development have been taken from the papers and books listed in the bibliography. . . . It was the writer's desire to present a rigorous treatment of Riemann surfaces, and it has been found in some cases that additions and changes to published proofs were necessary to make them valid. . . . The first chapter presents the necessary topological background. The second chapter is concerned with definitions related to a Riemann surface. The third chapter presents a proof of the Dirichlet principle which gives us the real part of our mapping function. The fourth chapter completes the solution of the mapping problem. *M. Heins* (Providence, R. I.).

Strebel, Kurt. Eine Bemerkung zur Hebbarkeit des Randes einer Riemannschen Fläche. Comment. Math. Helv. **23**, 350–352 (1949).

The present note complements a result of L. Sario [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); these Rev. **10**, 365]. It is shown with the aid of parallel slit mappings that a necessary condition that the boundary of a subregion of a closed Riemann surface be relatively removable is that it have zero two-dimensional measure.
M. Heins (Providence, R. I.).

Sario, Leo. Sur le problème du type des surfaces de Riemann. C. R. Acad. Sci. Paris **229**, 1109–1111 (1949).

The following theorem is proved with the aid of the Koebe $\frac{1}{4}$-theorem. If $\{F_{2n}\}$ is an exhaustion of a simply-connected Riemann surface F (F_{2n} simply-connected and relatively compact, $\bar{F}_{2n}\subset F_{2n+2}$) there exists a refinement $\{F_n\}$ with the property that the product of the modules of $F_{n+1}-\bar{F}_n$ is convergent. Thereby problem 1b, p. 76 of the author's thesis [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); these Rev. **10**, 365] receives a negative answer. *M. Heins* (Providence, R. I.).

Sario, Leo. Existence des fonctions d'allure donnée sur une surface de Riemann arbitraire. C. R. Acad. Sci. Paris **229**, 1293–1295 (1949).

Methods are described for establishing the existence of harmonic functions with assigned singularities on arbitrary Riemann surfaces. These methods circumvent the restrictions encountered in Nevanlinna's construction [Comment. Math. Helv. **22**, 302–316 (1949); these Rev. **10**, 525].
M. Heins (Providence, R. I.).

Sario, Leo. Quelques propriétés à la frontière se rattachant à la classification des surfaces de Riemann. C. R. Acad. Sci. Paris **230**, 42–44 (1950).

Announcement of results concerning the existence on a Riemann surface of harmonic (analytic) functions which are bounded or have bounded Dirichlet integral.
M. Heins (Providence, R. I.).

Sario, Leo. Existence des intégrales abéliennes sur les surfaces de Riemann arbitraires. C. R. Acad. Sci. Paris **230**, 168–170 (1950).

The results of the two papers reviewed above are applied to construct a theory of Abelian integrals on arbitrary Riemann surfaces. *M. Heins* (Providence, R. I.).

Sario, Leo. Questions d'existence au voisinage de la frontière d'une surface de Riemann. C. R. Acad. Sci. Paris **230**, 269–271 (1950).

Various criteria for parabolic type [Nevanlinna, Ahlfors, Laasonen] are related to a condition involving the divergence of the product of exhaustion modules [cf. the author's thesis, reference in the 4th preceding review].
M. Heins (Providence, R. I.).

Uskila, Leo. Über die Existenz der beschränkten automorphen Funktionen. Ark. Mat. 1, 1–11 (1949).

This paper is concerned with the existence or non-existence of bounded automorphic functions possessing a given group of linear substitutions with an orthogonal circle. A necessary and sufficient criterion for the existence of such functions is given in the case in which the group is symmetric, of genus zero, and does not contain elliptic substitutions. *Z. Nehari* (St. Louis, Mo.).

Ritt, J. F. Abel's theorem and a generalization of one-parameter groups. Trans. Amer. Math. Soc. 67, 491–497 (1949).

The author starts with the observation that $p+1$ points $b_1, \cdots, b_{p+1}$ determine in general uniquely p points $c_1, \cdots, c_p$, their "product" $\{b_1, \cdots, b_{p+1}\}$, by means of Abel's theorem $b_1+\cdots+b_{p+1}-(p+1)a\sim c_1+\cdots+c_p-pa$ with respect to the reference point a on a Riemann surface of genus p. Thus, $f_i(x_1, \cdots, x_{p+1})$ shall denote p symmetric functions which are analytic in a neighborhood of zero, P_i represent the p elementary symmetric functions of degree i for p variables. It is then assumed that f_i reduces to the function P_i of $x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{p+1}$ if $x_j=0$. The p relations $P_i(y_1, \cdots, y_p)=f_i(x_1, \cdots, x_{p+1})$ determine in a suitable neighborhood of the origin of the x-space a set $y_1, \cdots, y_p$ which is termed the product $\{x_1, \cdots, x_{p+1}\}$. Then the associativity condition $\{\{x_1, \cdots, x_{p+1}\}, x_{p+2}\}=\{x_1, \{x_2, \cdots, x_{p+2}\}\}$ is studied by means of a careful investigation of power series expansions and Jacobians. Associativity turns out to be equivalent to the existence of integrals $\Phi_i(z)\sim d_iz^i$ with $d_i\neq 0$ which are analytic for $z=0$, such that

$$\sum_{j=1}^{p}\Phi_i(y_j)=\sum_{j=1}^{p+1}\Phi_i(x_j), \qquad 1\leqq i\leqq p,$$

O. F. G. Schilling (Chicago, Ill.).

Schoeneberg, Bruno. Multiplikative Gruppen algebraischer Funktionen. Abh. Math. Sem. Univ. Hamburg 16, 136–139 (1949).

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Let N be an integer larger than 2. The author exhibits explicit generating elements of the field of algebraic functions which belongs to the principal modular group $\Gamma(N)$ and thus extends an important result of Hecke. His proof is based on Hecke's and Hurwitz's well-known result in which the Nth division values of the Weierstrass $\wp$-function, all of which have dimension -2, are identified as generating elements of the linear space of the Eisenstein series of dimension -2 belonging to the group $\Gamma(N)$. There are $\sigma-1$ independent such forms, where σ denotes the number of rational vertices of the canonical fundamental domain of $\Gamma(N)$. The set of vertices S_i has the following special property: all functions of the form $\sum_{i=1}^{\sigma}\nu_iS_i$, with integers ν_i satisfying $\sum_{i=1}^{\sigma}\nu_i=0$, form a free group with $\sigma-1$ free generators. Thus there are $\sigma-1$ functions $g_i=\mu_iS_i-\mu_iS_\sigma$ whose μ_ith roots are multiplicative radical functions with μ_ith roots of unity for multipliers relative to the surface of $\Gamma(N)$ and a pole and zero of order 1 at S_σ and S_i, respectively. Therefore this special property together with Hecke's theorem on the behaviour of Eisenstein series at the rational vertices of the fundamental domain implies the existence of the field generators. Finally the result is extended to arbitrary congruence groups $\Gamma_1(N)$ modulo N by combining suitable transforms of the fundamental domain of $\Gamma(N)$ and considering appropriate Nth division values of the $\wp$-function so as to obtain the right multipliers with respect to $\Gamma_1(N)$. *O. F. G. Schilling* (Chicago, Ill.).

Baganas, Nicolas. Sur les valeurs algébriques d'une fonction algébroïde et les intégrales pseudo-abéliennes. Ann. Sci. École Norm. Sup. (3) 66, 161–208 (1949).

Les fonctions algébroïdes $u=u(z)$ envisagées par l'auteur sont d'ordre fini: u de degré m est définie par

$$f_0(z)u^m+f_1(z)u^{m-1}+\cdots+f_m(z)=0$$

où les $f_j(z)$ sont des fonctions entières d'ordre fini. Par les méthodes élémentaires de la théorie des fonctions entières, l'auteur généralise le théorème de Borel sur les sommes de produits d'exponentielles par des fonctions entières et en déduit son théorème fondamental qui contient (pour l'ordre fini) des résultats antérieurs de Remoundos et de Montel: si $u(z)$ est de degré m et d'ordre fini, si $b(z)$ est une fonction algébrique de degré à branches distinctes et si $u(z)$ ne prend qu'un nombre fini de fois pour un même z les valeurs de $b(z)$, on a $n\leqq 2m$. Cette limite $2m$ peut être atteinte; les cas où $n<2m$ sont aussi étudiés. La théorie des combinaisons exceptionnelles de Montel [voir Leçons sur les familles normales, Gauthier-Villars, Paris, 1927, chapitre X] est étendue. Des extensions de ces résultats sont déduites de cet autre théorème général: l'ordre de la dérivée logarithmique de $u(z)$ (d'ordre fini) est au plus égal à l'ordre réel des zéros et à celui des pôles de la dérivée $u'(z)$. Diverses propositions relatives aux valeurs quasi-exceptionnelles (au sens de Valiron) sont aussi obtenues. Partant ensuite d'un problème posé par Montel [J. Math. Pures Appl. (9) 20, 305–324 (1941); ces Rev. 4, 7] l'auteur résout le suivant: étant donnée une fonction algébrique $b(z)$, trouver par un nombre fini d'opérations algébriques, les fonctions algébriques exceptionnelles pour $b(z)$, dépourvues de pôles à distance finie et dont les branches forment un seul système circulaire à l'infini. Partant de la remarque déjà utilisée plus haut: si une fonction algébroïde $G(z)$ d'ordre fini n'a qu'un nombre fini de zéros, de pôles et de points critiques, sa dérivée logarithmique G'/G est algébrique, donc log G est une intégrale abélienne, l'auteur étudie ce qu'il appelle intégrales pseudo abéliennes. Ce sont des intégrales abéliennes relatives à une courbe algébrique Γ qui s'expriment par une somme d'un nombre fini de logarithmes de fonctions algébroïdes ou algébriques uniformes sur la surface de Riemann de Γ. Il montre que les algébroïdes ainsi introduites sont nécessairement d'ordre fini et ont un nombre fini de pôles et de zéros. Il applique ce résultat à l'étude de certaines intégrales hyperelliptiques. *G. Valiron* (Paris).

Conforto, Fabio. Sulla nozione di corpi equivalenti e di corpi coincidenti nella teoria delle funzioni quasi abeliane. Rend. Sem. Mat. Univ. Padova 18, 292–310 (1949).

L'autore chiama equivalenti due corpi di funzioni abeliane o quasi-abeliane (per la definizione e la teoria delle funzioni quasi-abeliane vedasi F. Severi [Funzioni quasi abeliane, Pontificiae Academiae Scientiarum Scripta Varia, v. 4, Città del Vaticano, 1947; questi Rev. 9, 578]) quando si passi da un corpo all'altro mediante una sostituzione lineare sulle variabili indipendenti. Mentre nel caso abeliano l'equivalenza trae seco la coincidenza dei due corpi quando i periodi siano uguali, così non è nel caso quasi-abeliano. L'autore dà, per questo secondo caso, come condizione necessaria e sufficiente per la coincidenza che la predetta sostituzione lineare possa interpretarsi come una trasformazione birazionale in sè della varietà di Picard associata ad uno dei corpi.

E. Martinelli (Genova).

***Whittaker, J.-M. Sur les Séries de Base de Polynomes Quelconques. Avec la collaboration de C. Gattegno.** Gauthier-Villars, Paris, 1949. vi+85 pp. 1000 francs.

This monograph is a self contained exposition of the representation of analytic functions by series of the form $\sum A_k p_k(z)$, where the set of polynomials $\{p_0, p_1, \cdots\}$ forms a basis for the class of polynomials, as developed by the author and his students, and begun by him [Interpolatory Function Theory, Cambridge University Press, 1935]. The ground covered is roughly the same as that contained in a recent, but somewhat inaccessible, publication [Series of Polynomials, Fouad I University, Faculty of Science, Cairo, 1943; these Rev. **8**, 454] based upon the author's lectures; the present treatment is much fuller, and also contains results obtained since 1943, dealing with the inverse of a basic set, the product of two sets, and the connection between the location of the zeros of the polynomials $\{p_n\}$ and their representation properties, most of which have been published elsewhere. *R. C. Buck* (Madison, Wis.).

Martin, Yves. Sur les séries d'interpolation. Ann. Sci. École Norm. Sup. (3) **66**, 311–366 (1949).

Let $\{\lambda_\nu\}_0^\infty$ be a sequence of nonnegative real numbers such that: (1) $\{\lambda_\nu\}\varepsilon\uparrow$, $\lim_{\nu\to\infty}\lambda_\nu=\infty$, $\sum_0^\infty\lambda_\nu^{-1}=\infty$; (2) $\{\lambda_\nu\}\varepsilon\uparrow$, $\sum_0^\infty\lambda_\nu^{-1}<\infty$; (3) $\{\lambda_\nu\}\varepsilon\downarrow$, $\sum_0^\infty\lambda_\nu<\infty$. A very detailed study is made in these cases of the series (*) $\sum_{n=0}^\infty a_nP_n(z)$ where $P_n(z)=\prod_{\nu=1}^n(1-z\lambda_\nu^{-1})$. If the series (*) converges for any z not in $\{\lambda_\nu\}_0^\infty$ then its region of convergence is: a half plane $\Re z>S$ in case (1); the entire z-plane in case (2); a circle $|z|<R$ in case (3). The region of absolute convergence coincides with the region of convergence in case (3). In case (1) the region of absolute convergence is a half plane $\Re z>S'$. An upper bound for $S'-S$ is given in terms of the density of $\{\lambda_\nu\}_0^\infty$. In case (1) a study is made of conditions which are necessary and also of conditions which are sufficient if $f(z)$ is to have a representation $f(z)=\sum_0^\infty a_nP_n(z)$. Both sets of conditions are of the form $|f(z)|\leqq AF(r,\theta)$ where $F(r,\theta)$ is determined by $\{\lambda_\nu\}_0^\infty$. These sets of conditions differ only slightly. In this case the behaviour of the series (*) is closely related to that of the generalized Dirichlet series $\sum_{n=0}^\infty a_ne^{-S(n)z}$ where $S(n)=\sum_{\nu=0}^n\lambda_\nu^{-1}$. In case (3) the behaviour of the series (*) on its circle of convergence is shown to resemble closely that of a power series.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Mergelyan, S. N. On best approximations in a complex region. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 202–204 (1949). (Russian)

A brief resumé of the results of a doctoral dissertation. [Reviewer's note: these results are contained, substantially, in four earlier publications of the author [Doklady Akad. Nauk SSSR (N.S.) **61**, 981–983 (1948); **62**, 23–26, 163–166 (1948); **63**, 220 (1948); these Rev. **10**, 242, 243.]

I. M. Sheffer (State College, Pa.).

Walsh, J. L., and Sewell, W. E. On interpolation to an analytic function in equidistant points: Problem β. Bull. Amer. Math. Soc. **55**, 1177–1180 (1949).

In extending the results of an earlier paper [Trans. Amer. Math. Soc. **49**, 229–257 (1941)] the authors prove the following theorem. Let the function $f(z)$ be analytic and let its pth derivative satisfy a Lipschitz condition of order α in the annulus $\rho>|z|>1/\rho$. Let $p_n(z)=\sum_{k=-n}^n a_{n,k}z^k$ interpolate to $f(z)$ in the $(2n+1)$th roots of unity. Then, for $|z|=1$, $|f(z)-p_n(z)|\leqq M/\rho^n n^{p+\alpha}$, where M is independent of n and z. Approximation on more general point sets by rational functions with poles at prescribed points other than the origin are indicated as immediate generalizations of this theorem.

E. N. Nilson (Hartford, Conn.).

Galbraith, A. S., Seidel, W., and Walsh, J. L. On the growth of derivatives of functions omitting two values. Trans. Amer. Math. Soc. **67**, 320–326 (1949).

Let $w=f(z)$ be regular in $|z|<1$ and omit two finite values. Let $D_p(w)$ be the radius of p-valence in the Riemann configuration of $f(z)$ [Seidel and Walsh, same Trans. **52**, 128–216 (1942), chapter IV; these Rev. **4**, 215]. Let z_n be a sequence of points in $|z|<1$, $w_n=f(z_n)$ and suppose that as $n\to\infty$, $w_n\to\infty$ and (*) $\{|w_n|(\log|w_n|)\}^pD_p(w_n)\to0$. Then $(1-|z_n|^2)^k|f^{(k)}(z_n)|\to0$, $k=0, 1, \cdots, p$. This sharpens a previous result of two of the authors [loc. cit.] with $(2^p-1)(1+\epsilon)$ instead of p in the index in (*).

W. K. Hayman (Stanford University, Calif.).

Moppert, Karl-Felix. Über Relationen zwischen m- und p-Funktionen. Verh. Naturforsch. Ges. Basel **60**, 61–76 (1949).

Proof of results announced earlier [Comment. Math. Helv. **23**, 174–176 (1949); these Rev. **11**, 93] concerning rational relations $m=R(p)$ between p functions regular in $|z|<1$ and $p(z)\neq0$, 1 and m functions, which take 0, 1 only with multiplicity divisible by v_0, v_1, respectively, in $|z|<1$. Only the cases $v_0=3$, $v_1=2$; $v_0=4$, $v_1=2$; $v_0=v_1=3$; $v_0=2$, $v_1=\infty$; $v_0=v_1=\infty$, or vice versa give rise to such rational relations. In these cases the known bounds for p functions (Schottky's theorem) are extended to bounds for m functions of similar accuracy. *W. K. Hayman.*

Valiron, Georges. Sur les valeurs déficientes des fonctions méromorphes d'ordre nul. C. R. Acad. Sci. Paris **230**, 40–42 (1950).

The author proves that a meromorphic function of order zero possesses at most one value deficient in the sense of Nevanlinna. Sharper results are obtained under stronger restrictions on the growth of the characteristic.

W. K. Hayman (Stanford University, Calif.).

***Bergmann, Stefan. Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques.** Mémor. Sci. Math., no. 106. Gauthier-Villars, Paris, 1947. 63 pp.

This book, which was originally to have appeared in 1940, was republished in 1941 by Interscience Publishers, New York, and reviewed in these Rev. **2**, 359.

***Bergmann, Stefan. Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes.** Mémor. Sci. Math., no. 108. Gauthier-Villars, Paris, 1948. 80 pp.

Vorliegendes Heft ist die Fortsetzung eines Heftes des gleichen Verfassers [siehe vorstehendes Referat] und setzt dessen Kenntnis voraus. Es handelt sich um eine Zusammenstellung der bisher bewiesenen Tatsachen und der Fragestellungen der sogenannten Bergmann'schen Abbildungstheorie, deren Hauptziel ist, Verallgemeinerungen für die beiden grundlegenden Sätze der Theorie der konformen Abbildungen: den Abbildungssatz von Riemann-Poincaré und das Schwarz'sche Lemma aufzustellen [siehe auch die kurze Darstellung in Behnke und Thullen, Theorie der Funktionen mehrerer komplexer Veränderlichen, Springer, Berlin, 1934]. Die wichtigsten Hilfsmittel sind die sogenannten Minimalfunktionen, die Kernfunktion und die durch sie definierte, gegenüber analytischen Abbildungen

invariante Metrik $G_{\mathfrak{B}}$ des vorgegebenen Bereiches $\mathfrak{B}$. Die Aussagen beschränken sich auf den Raum R_4 zweier komplexer Veränderlichen.

Im ersten Kapitel wird das Verhalten der Kernfunktion eines gegebenen Bereiches $\mathfrak{B}$ bei Annäherung an einen Randpunkt untersucht, in dessen Umgebung $\mathfrak{B}$ pseudokonvex ist, wobei gewisse zusätzliche Voraussetzungen gemacht werden, welche je nach ihrer Art eine Klasseneinteilung der Randpunkte bestimmen. Für einen Randpunkt Q der ersten Klasse (in diesem Falle gibt es eine analytische Fläche, die mit dem abgeschlossenen Bereich $\mathfrak{B}$ nur Q gemeinsam hat; ausserdem wird der Rand von $\mathfrak{B}$ zweimal stetig differentierbar vorausgesetzt) gelingt es, explizite Grenzausdrücke für die (unendlich werdende) Kernfunktion ihre ersten Ableitungen, für ds^2 und die Invarianten I und R der erwähnten Metrik $G_{\mathfrak{B}}$ anzugeben. Die Grenzausdrücke und Abschätzungen in diesem und den andern Fällen werden dadurch gewonnen, dass man den Bereich $\mathfrak{B}$ in der Umgebung von Q durch leicht zu übersehende "innere" und "äussere" Vergleichsbereiche ersetzt, deren Art von der Struktur des Randes abhängt.

Im zweiten Kapitel wird die Theorie der Repräsentantenbereiche entwickelt. Zwei Bereiche gehören zu einer Klasse (im engeren Sinne), falls sie durch eine in einem inneren Punkte normierte analytische Abbildung ineinander übergeführt werden können. Jeder Klasse kann eindeutig ein Repräsentantenbereich und jedem Bereich der Klasse eindeutig ein normiertes Funktionspaar, das ihn auf den Repräsentantenbereich abbildet, zugeordnet werden. Der Cartansche Satz über Bereiche mit einer unendlichen Automorphismengruppe mit innerem Fixpunkt und die Abbildungssätze für Kreiskörper und verallgemeinerte Cartansche Kreiskörper können mit Hilfe der Theorie der Repräsentantenbereiche abgeleitet werden. Es werden auch Bereiche mit allgemeineren Automorphismengruppen ("homogene" und "symmetrische" Bereiche) betrachtet und schliesslich Automorphismen, die nicht notwendig einen inneren Fixpunkt aufweisen. Hingewiesen sei noch auf den Begriff der "Minimalbereiche," die durch gewisse Minimalforderungen definiert sind, z.B. Bereiche die unter einer Menge aufeinander normiert abbildbarer Bereiche das kleinste Volumen haben.

Im dritten Kapitel handelt es sich um die Aufstellung von Schranken für die Variation gewisser mit der euklidischen Metrik verbundenen Grössen (Kurvenlängen, Flächeninhalte, usw.) bei analytischen Abbildungen des gegebenen Bereiches $\mathfrak{B}$ auf Bereiche $\mathfrak{G}$, die alle in einem festen Bereich $\mathfrak{A}$ enthalten seien. Wichtige Sonderfälle sind: $\mathfrak{A}=\mathfrak{B}$ (jedes $\mathfrak{G}$ Teilbereich von $\mathfrak{B}$); oder falls $\mathfrak{A}$ den ganzen Raum ausfüllt mit Ausnahme gewisser Ebenen oder gewisser Hyperebenen.

In einem letzten Kapitel wird die Bedeutung der Kernfunktion am Beispiel der konformen Abbildungen der klassischen Theorie einer Veränderlichen gezeigt, indem nachgewiesen wird, dass die wichtigsten einem (nicht notwendig einfach zusammenhängenden) Bereiche zugeordneten harmonischen Funktionen (Greensche und Neumannsche Funktion, usw.) durch die Kernfunktion ausgedrückt werden können.

P. Thullen (Panamá).

Cartan, Henri. Sur un cas de prolongement analytique pour les fonctions de plusieurs variables complexes. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **61**, 6 pp. (1949).

Verf. geht von dem von Myrberg gestellten Problem aus, eine gewisse Aussage über Funktionen zweier komplexer Veränderlichen auf solche von n Veränderlichen zu verallgemeinern und beweist zunächst den folgenden Satz. In dem Raume der n komplexen Veränderlichen $x_1, \cdots, x_p; y_1, \cdots, y_q$ $(p+q=n)$ habe ein Bereich D die Eigenschaft, dass mit einem Punkt (x_i, y_j) aus D stets auch $(\lambda_i x_i, \mu_j y_j)$ zu D gehört, wobei λ_i und μ_j $(i=1, \cdots, p;\ j=1, \cdots, q)$ feste gegebene Konstanten sind mit $|\lambda_i|>1$, $|\mu_j|<1$. Falls dann noch D einen Punkt $(x_i=a_i, y_j=0)$ und zugleich einen Punkt $(x_i=0, y_j=b_j)$ enthält, so ist jede in D analytische Funktion auch im Anfangspunkt $(0, \cdots, 0)$ regulär. Hieraus folgt dann die gesuchte Aussage, dass nämlich eine in D analytische, dort der Funktionalgleichung $f(\lambda_i x_i, \mu_j y_j)=f(x_i, y_j)$ genügende Funktion notwendig eine Konstante ist, abgesehen von leicht anzugebenden trivialen Ausnahmefällen.

P. Thullen (Panamá).

Myrberg, P. J. Réflexions sur la note précédente. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **62**, 4 pp. (1949).

Es wird auf den Zusammenhang des obigen Cartan'schen Satzes [siehe die vorgehende Arbeit] mit der Singularitätentheorie automorpher Funktionen hingewiesen. Unter anderm wird die folgende Frage aufgeworfen. Aus dem Cartan'schen Satze folgt im Falle dreier komplexer Veränderlichen, dass eine Funktion $f(x, y, z)$ welche der Gleichung

$$f(ax, by, cz)=f(x, y, z), \qquad |a|>1,\ |b|>1,\ |c|<1,$$

genügt, entweder auf der ganzen Hyperebene $z=0$ oder in allen Punkten der zweidimensionalen Fläche $x=y=0$ singulär ist. Im letzteren Falle muss es eine die Fläche $x=y=0$ enthaltende und für $f(x, y, z)$ singuläre vierdimensionale Hyperfläche geben. Verf. vermutet nun, dass diese Hyperfläche entweder mit $x=0$ oder mit $y=0$ identisch sein muss (bis auf explizit anzugebende Ausnahmen). Aus der Lösung dieses Problems könnte dann eine allgemeinen Aussage über die Singularitäten automorpher Funktionen, die zu einer unstetigen allgemeinen Gruppe linearer Transformationen gehören, gewonnen werden. *P. Thullen* (Panamá).

Tsuji, Masatsugu. On the boundary value of a bounded analytic function of several complex variables. Proc. Japan Acad. **21** (1945), 308–312 (1949).

Enoncé et démonstration de la propriété suivante: si $f(z, w)$ est holomorphe et bornée dans $[|z|<1, |w|<1]$, alors: (I) $\lim f(z, w)=f(e^{i\theta}, e^{i\varphi})$ existe presque partout sur l'arête S du dicylindre, quand $z\rightarrow e^{i\theta}$, $w\rightarrow e^{i\varphi}$, non tangentiellement à $|z|=1$ et à $|w|=1$ respectivement; (II) si la limite $f(e^{i\theta}, e^{i\varphi})$ s'annule sur un ensemble de mesure positive sur S, alors $f(z, w)\equiv 0$; (III) si $f(z, w)\not\equiv 0$, $\log|f(e^{i\theta}, e^{i\varphi})|$ est intégrable sur S. *P. Lelong* (Lille).

Wintner, Aurel. On implicit analytic systems. Comment. Math. Helv. **23**, 294–302 (1949).

Soit D un domaine connexe, contenant l'origine, dans l'espace de p variables complexes $z_1, \cdots, z_p$; et Δ un domaine connexe, contenant l'origine, dans l'espace de n variables complexes $w_1, \cdots, w_n$. Soit $(f_j(z, w))$ un système de n fonctions analytiques pour $z=(z_1, \cdots, z_p)\epsilon D$ et $w=(w_1, \cdots, w_n)\epsilon\Delta$; on suppose $f_j(0, 0)=0$, et que les valeurs des f_j sont dans Δ. L'auteur, se plaçant dans le cas où D est $|z_1|<1, \cdots, |z_p|<1$, et où Δ est $|w_1|<1, \cdots, |w_n|<1$, démontre le théorème suivant (qui vaut pour D et Δ quelconques): si les dérivées partielles $\partial f_j/\partial w_k$ sont nulles à l'origine, il existe une application analytique et une seule $z\rightarrow\varphi(z)$ de Δ dans Δ, telle que $\varphi(0)=0$ et $\varphi(z)=f(z, \varphi(z))$ identiquement dans Δ. En fait, l'existence ne nécessite pas

d'hypothèse sur les $\partial f_j(0, 0)/\partial w_k$, et se prouve par itération, en utilisant les propriétés classiques des familles bornées de fonctions analytiques; quant à l'unicité, elle est assurée globalement dès qu'elle est assurée au voisinage de l'origine, ce qui est notamment le cas si la matrice des $\partial f_j(0, 0)/\partial w_k$ n'admet pas la valeur propre 1. *H. Cartan* (Paris).

***Heyda, James Francis. Uniqueness Properties Over Sets of Positive Linear Measure of Functions of a Complex Variable.** Abstract of a Thesis, University of Illinois, 1940. ii+12 pp.

Let K be a bounded region in the (x, y) plane and let $\bar{K}$ be its closure. Let E be a closed set having a positive planar measure and $C(E)$ its complement with respect to $\bar{K}$. The author considers the class of functions $f(w)$ defined for $w=u+iv\epsilon E$ by

$$f(w)=\int\int_{C(E)} q(x, y)(x+iy-w)^{-1}dxdy,$$

where $q(x, y)$ runs through the class of functions continuous in $\bar{K}$ and vanishing in E. This class is closely related to general monogenic functions as defined and studied by W. J. Trjitzinsky [Ann. Sci. École Norm. Sup. (3) **55**, 119–191 (1938); Acta Math. **70**, 63–163 (1938); **78**, 97–192 (1946); these Rev. **7**, 381]. Using results of Trjitzinsky the author gives conditions under which each function $f(w)$ of the class defined above which vanishes in a certain linear set P of positive linear measure will vanish also in a certain larger linear set P^* such that $P^*\supset P$ and $|P^*|>|P|$. These conditions are too complicated to be reproduced here.
S. Agmon (Houston, Tex.).

***Helton, Floyd Franklin. Quasi Analyticity Related to Sets of Positive Measure for Functions of a Complex Variable.** Abstract of a Thesis, University of Illinois, 1946. ii+9 pp.

The author treats a problem similar to that considered above. The only difference is that P and P^* are now sets of positive planar measure. Using results of Trjitzinsky [same as above] conditions for the existence of the larger set P^* are given. *S. Agmon* (Houston, Tex.).

***Evans, Willie Buell. Uniqueness Properties of General Monogenic Functions.** Abstract of a Thesis, University of Illinois, 1950. i+3 pp.

The author announces extensions of results of Trjitzinsky [Ann. Sci. École Norm. Sup. (3) **55**, 119–191 (1938)].
R. P. Boas, Jr. (Providence, R. I.).

***Mathews, Charles Willard, Jr. Cauchy Type Double Integral Representation for Functions of a Complex Variable.** Abstract of a Thesis, University of Illinois, 1947. i+4 pp.

The author studies a representation given by Trjitzinsky [J. Math. Pures Appl. (9) **25** (1946), 347–395 (1947); these Rev. **9**, 418] for a class of nonanalytic functions and states conditions under which it is equivalent to a regular Fredholm integral equation. *R. P. Boas, Jr.* (Providence, R. I.).

Theory of Series

Sengupta, H. M. On rearrangements of series. Proc. Amer. Math. Soc. **1**, 71–75 (1950).

In the set E of all rearrangements $x=(x_1, x_2, \cdots)$ of the integers $1, 2, \cdots$ the distance

$$\rho(x, y)=\sum 2^{-n}|x_n-y_n|(1+(x_n-y_n))^{-1}$$

is introduced. For a given real series $\sum_{n=1}^{\infty}c(n)$, the author investigates the set A of all x for which

$$\varphi(x)=\sup_n \sum_{r=1}^{n} c(x_r)<+\infty$$

[for the category of A, see Agnew, Bull. Amer. Math. Soc. **46**, 797–799 (1940); these Rev. **2**, 89]. His main theorem is that A is an F_σ in E. [The reviewer remarks that this is an immediate consequence of the fact that $\varphi(x)$ is lower semicontinuous.] *G. G. Lorentz* (Toronto, Ont.).

Bagchi, Hari Das. Note on a class of infinite Riemannian integrals. Bull. Calcutta Math. Soc. **41**, 103–112 (1949).

The de la Vallée Poussin test for absolute and uniform convergence of an integral over an infinite range is applied in several special cases. *J. F. Randolph.*

Szász, Otto. Summation of slowly convergent series. J. Math. Physics **28**, 272–279 (1950).

Let $p_0>0$, $p_n\geqq 0$, $P_n=p_0+p_1+\cdots+p_n$ and $P_n\to\infty$. When s_n is an increasing sequence of positive numbers converging to s, s_n is a better approximation to s than the Riesz transform (1) $t_n=P_n^{-1}(p_0s_0+p_1s_1+\cdots+p_ns_n)$. This observation suggests that the inverse transformation (2) $t_n=s_n+P_{n-1}p_n^{-1}(s_n-s_{n-1})$ may be useful for evaluation of sequences s_n of partial sums of series $\sum u_n$ of positive terms and of series $\sum u_n$ that converge slowly. Instead of (2), the author writes (3) $t_n=s_n+\rho_nu_n$, where $\rho_n>0$. In case $u_{n+1}/u_n=1-a/n+\gamma_{n-1}/n$, where $a>1$ and $\gamma_n\to 0$, it is shown that if $\rho_n=(n+1)/(a-1)$ then t_n converges more rapidly than s_n in the sense that $|t_n-s|/|s_n-s|\to 0$. If ρ_n is such that $s_n+\rho u_n$ is a better approximation to s than s_n is whenever $u_n>0$ and $\sum u_n=s$, then $\sum\rho_n^{-1}=\infty$. Finally the author assumes that $\sum\rho_n^{-1}=\infty$ and determines a class of convergent series depending on $\rho_0, \rho_1, \cdots$ for each of which t_n converges more rapidly than s_n. *R. P. Agnew.*

Leng, Sen-ming. Note on Cauchy's limit theorem. Amer. Math. Monthly **57**, 28–31 (1950).

The Silverman-Toeplitz theorem on regularity of linear transformations of the type $t_n=\sum_{k=0}^{\infty}a_{nk}s_k$ is modified to cover real nonlinear transformations of real sequences. Let $f_n(s_1, s_2, \cdots)$ be a sequence of real functions such that $t_n=f_n(s_1, s_2, \cdots)$ exists whenever s_n is a convergent sequence. Let $f_n(s_1, s_2, \cdots)-f_n(s_1', s_2', \cdots)\to 0$ whenever s_n converges and $s_n'=s_n$ except for one value of n. Let inf $s_k\leqq$ lim inf t_n and lim sup $t_n\leqq$ sup s_k whenever s_k converges. Then $t_n\to s$ whenever $s_n\to s$. Moreover, the above conditions on f are necessary. The author's formulation is slightly more complex than the above, involving only transforms of sequences of the form $s, s, \cdots, s_k, s_{k+1}, \cdots$, where s_n is a single fixed sequence converging to s. *R. P. Agnew* (Ithaca, N. Y.).

Knopp, K., und Lorentz, G. G. Beiträge zur absoluten Limitierung. Arch. Math. **2**, 10–16 (1949).

A sequence s_n is said to be absolutely convergent when it has bounded variation, that is, $\sum|s_n-s_{n-1}|<\infty$. The sequence-to-sequence transformation

$$(1)\qquad \sigma_n=\sum_{k=0}^{\infty}a_{nk}s_k, \qquad n=0, 1, 2, \cdots,$$

is absolutely conservative if absolute convergence of s_n implies absolute convergence of σ_n, and is absolutely regular if absolute convergence of s_n and $s_n\to s$ imply absolute convergence of σ_n and $\sigma_n\to s$. Similarly, the series-to-series transformation

$$(2)\qquad \alpha_n=\sum_{k=0}^{\infty}b_{nk}a_k, \qquad n=0, 1, 2, \cdots,$$

is absolutely conservative if $\sum|a_n|<\infty$ implies $\sum|\alpha_n|<\infty$, and is absolutely regular if $\sum|a_n|<\infty$ and $\sum a_n=s$ imply $\sum|\alpha_n|<\infty$ and $\sum\alpha_n=s$. It is shown that (2) is absolutely conservative if and only if there is a constant M such that

$$(3)\qquad \sum_{n=0}^{\infty}|b_{nk}|\leqq M, \qquad k=0,1,2,\cdots,$$

and that (2) is absolutely regular if and only if (3) holds and

$$(4)\qquad \sum_{n=0}^{\infty}b_{nk}=1, \qquad k=0,1,2,\cdots.$$

This result is used to obtain the conditions of Mears [Ann. of Math. (2) 38, 594–601 (1937)] that (1) be absolutely conservative, and that (1) be absolutely regular. Applications are given to the Hurwitz-Silverman-Hausdorff methods H which have the form (1) where $a_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k$, $0\leqq k\leqq n$, and $a_{nk}=0$ when $k>n$. If H is conservative (which happens only when μ_k is a moment sequence) then H is absolutely conservative; and if H is regular then H is absolutely regular. This leads immediately to relations of inclusion and equivalence involving absolute summabilities by methods of Euler-Knopp, Cesaro and Hölder; some of these relations have previously been proved by less efficient methods.

Simple conditions analogous to (3) and (4) imply that the integral transformation

$$(5)\qquad \alpha(x)=\int_0^\infty b(x,t)ds(t), \qquad x>0,$$

is such that $\int|\alpha(x)|dx<\infty$ whenever $s(t)$ has bounded variation over $t>0$. Use of (5) gives such relations, at least some of which are known, among absolute summabilities as $|C_k|\subset A$ (A = power-series method), $|C_k|\subset|C_r|$ when $0<k<r$ for integral transformations, $|E_k|\subset B$ (B = Borel method), and $|B_k|\subset|B_r|$ when $0\leqq k<r$. *R. P. Agnew.*

Karamata, J. Über die Beziehung zwischen dem Bernsteinschen und Cesàroschen Limitierungsverfahren. Math. Z. 52, 305–306 (1949).

Let $\sum u_n$ be a series with partial sums s_n, and let

$$B_n=\sum_{k=0}^{n}(\cos k\pi/(2n+1))u_k, \quad \sigma_n=(n+1)^{-1}\sum_{k=0}^{n}\tfrac{1}{2}(s_{k-1}+s_k).$$

The author [Rec. Math. [Mat. Sbornik] N.S. 21(63), 13–24 (1947); these Rev. 9, 140] has previously shown that if one of $\lim B_n$ and $\lim \sigma_n$ exists, then both exist and they are equal. This note gives a simple direct proof of this result, which shows that Bernstein summability of order $\frac{1}{2}$ is stronger than C_1. *R. P. Agnew* (Ithaca, N. Y.).

Fuchs, W. H. J. On the "collective Hausdorff method." Proc. Amer. Math. Soc. 1, 66–70 (1950).

A sequence $\{s_n\}$ is said to be summable to S by the Hausdorff method $T\sim\mu_n$, if $\lim_{n\to\infty}t_n=S$, where $\{t_n\}$ is given by

$$\Delta^n t_0=\sum_{k=0}^{n}(-1)^k\binom{n}{k}t_k=\mu_n\Delta^n s_0, \qquad n=0,1,\cdots.$$

The sequence $\{s_n\}$ is summable by the "collective Hausdorff method $\mathfrak{H}$" if it is summable by any regular Hausdorff method. The author shows that there is no matrix method of summation that contains $\mathfrak{H}$. The method $\mathfrak{H}$ is regular and sums the sequences $\{(-c)^n\}$, $c>1$, and $\binom{n}{k}$, $k=1,2,\cdots$, to zero. Using these facts and the technique of rapidly increasing sequences, the author constructs a sequence that is not summable by a pre-assigned matrix A but is summable by $\mathfrak{H}$. The moment function of the Hausdorff method that sums this sequence has zeros in the right-half plane. It is not known whether the result is true when such methods are excluded from the "collective Hausdorff method."
H. G. Eggleston (Swansea).

Satô, Tokui. On an extension of the Carleman-Hukuhara theorem on asymptotic series. Mem. Fac. Sci. Kyûsyû Univ. A. 4, 29–31 (1949). (Esperanto)

Theorem of Carleman-Hukuhara: if a positive number α and the numbers a_n are given, then there exists an analytic function in the domain $|\arg z|<\alpha$; $|z|>1$ with the asymptotic expansion $\sum_{n=0}^{\infty}a_nz^{-n}$. The author proves: if the positive number $\alpha<\pi$ and the numbers a_{mn} are given, then there exists for each integer k a function which is analytic in the domain $|\mathfrak{J}(z)-2k\pi|<\alpha$; $|z|>1$, with the asymptotic expansion $\sum_{n=0}^{\infty}\varphi_n(z)\exp(-nz)$, where the analytic function $\varphi_n(z)$ possesses the asymptotic expansion $\sum_{k=0}^{\infty}a_{kn}z^{-k}$.
J. G. van der Corput (Amsterdam).

Cheng, Min-Teh. Some Tauberian theorems with application to multiple Fourier series. Ann. of Math. (2) 50, 763–776 (1949).

The object of the author is to prove theorems which are converse to the general Abelian theorems of Bochner-Hardy type [cf. Bochner and Chandrasekharan, Fourier Transforms, Princeton University Press, Princeton, N. J.; Oxford, 1949, pp. 48–49; these Rev. 11, 173]. The first step in that direction was taken by N. Wiener [J. London Math. Soc. 2, 118–123 (1927)] and the present paper aims at a generalization. The problem is not completely solved, but the following results are proved. (A) Let $\mu>0$, $0<\nu<\mu+\frac{1}{2}$; $T^{-\nu}\int_0^T|f(t)|dt<M$ for all $T>0$; $f(t)\geqq0$ for all $t>0$. Then the limit-relation

$$\lim R^{\nu-\mu}\int_0^\infty f(t)t^{-\mu}J_\mu(Rt)dt=l\nu\int_0^\infty t^{\nu-\mu-1}J_\mu(t)dt$$

as $R\to0$ or $R\to\infty$ implies

$$(*)\qquad \lim T^{-\nu}\int_0^T f(t)dt=l$$

as $T\to\infty$ or $T\to0$, respectively. Here $J_\mu(t)$ stands for the Bessel function of the first kind. In the case $\mu=\frac{1}{2}$ we have (B): let n be a positive integer, $0<\nu<n$, $T^{-\nu}\int_0^T|f(t)|dt<M$ for all $T>0$; $f(t)\geqq0$ for all $t>0$; then the limit-relation

$$\lim R^{\nu-n}\int_0^\infty f(t)(t^{-1}\sin Rt)^n dt=l\nu\int_0^\infty t^{\nu-n-1}\sin^n t\,dt$$

implies (*).

Applying the results (A) and (B) to the spherical summation of multiple Fourier series, the author derives a necessary and sufficient condition for the summability of the Fourier series of a function at a point in a neighborhood of which it is nonnegative. This is, however, a consequence of earlier results due to the reviewer [cf. Proc. London Math. Soc. (2) 50, 210–222 (1948), theorem IV, remark (iii), and theorem VII; these Rev. 10, 113]. *K. Chandrasekharan.*

Fourier Series and Generalizations, Integral Transforms

★Rogosinski, Werner. Fourier Series. Chelsea Publishing Company, New York, N. Y., 1950. vi+176 pp. $2.50.

A translation by Harvey Cohn and F. Steinhardt of Fouriersche Reihen [de Gruyter, Berlin, 1930]. The bibliography has been enlarged and an index added.

Winslow, A. M. A simplified method of differentiating and evaluating functions represented by Fourier series. Quart. Appl. Math. 7, 423–425 (1950).

When a function $f(x)$ satisfies sufficient conditions for the differentiation of its Fourier sine series on an interval $(0, a)$ except that $f(a)\neq 0$, the author points out that the sine series for the function $f(x)-xf(a)/a$ can be differentiated. This device gives the cosine series for $f'(x)$. Generalizations are discussed. *R. V. Churchill* (Ann Arbor, Mich.).

Yano, Shigeki. Notes on Fourier analysis. XV. On the absolute convergence of trigonometrical series. Tôhoku Math. J. (2) 1, 46–49 (1949).

The author proves the following theorem. If $\rho_n\geqq 0$, $\rho_n=O(1/n)$ and $\sum_{n=1}^\infty \rho_n=\infty$, then we have

$$\sum_{k=1}^{\infty}\rho_k|\cos(kx-\alpha_k)|\sim(2/\pi)\sum_{k=1}^{n}\rho_k$$

except in a set of α-capacity zero $(0<\alpha<1)$. The proof depends on a lemma of similar character for an exponential sum, and is related to results of Salem [Duke Math. J. 8, 317–334 (1941); these Rev. 2, 360]. *R. M. Redheffer.*

Yano, Sigeki. Notes on Fourier analysis. XVII. The integrated Lipschitz condition of a function and Fejér mean of Fourier series. Tôhoku Math. J. (2) 1, 50–56 (1949).

[Cf. the preceding review; no note XVI seems to have appeared.] Let $f(x)$ be a function belonging to Lip (α, p). The paper is concerned with the rate of convergence for the Fejér means σ_n of the Fourier series to f in the metric L^p. It is known that $M_p(f-\sigma_n)=O(n^{-\alpha})$, provided $0<\alpha<1$, $p\geqq 1$. By a counter example giving $M_1(f-\sigma_n)>Cn^{-1}\log n$, the author shows that the theorem fails for $\alpha=p=1$, and shows that even for an absolutely continuous f one may have $M_1(f-\sigma_n)>Cn^{-1}\log\log n$. If $\tilde{f}$ is the conjugate of f, and $\tilde{\sigma}_n$ the Fejér means of the conjugate series, then it is known that $M_1(f-\sigma_n)=O(n^{-\alpha})$ implies $M_1(\tilde{f}-\tilde{\sigma}_n)=O(n^{-\alpha})$, $0<\alpha<1$, $p\geqq 1$. The author shows that the theorem fails for $\alpha=1$, but proves that then $M_1(\tilde{f}-\tilde{\sigma}_n)=O(n^{-1}\log n)$. If $f\epsilon$Lip $(1, 1)$, a necessary and sufficient condition for $M_1(f-\sigma_n)=O(1/n)$ is $\tilde{f}\epsilon$Lip $(1, 1)$. *R. M. Redheffer.*

Civin, Paul. Approximation in Lip (α, p). Bull. Amer. Math. Soc. 55, 794–796 (1949).

Let L_p, $1<p<\infty$, denote the class of measurable functions of period 2π for which $(\int_{-\pi}^{\pi}|f(x)|^p dx)^{1/p}=M_p(f)<\infty$, and let Lip (α, p), $0<\alpha<\infty$, represent that subclass of L_p for which $M_p[f(x+h)-f(x)]=O(h^{-\alpha})$ as $h\to 0$. Using a method of M. Zamansky [C. R. Acad. Sci. Paris 224, 704–706 (1947); these Rev. 8, 457] the author proves the following theorem. If $f(x)\epsilon$Lip (α, p) and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order n such that $M_p(f-P_n)\leqq Kn^{-\alpha}$, then $M_p(P_n')\leqq A(1-\alpha)^{-1}n^{1-\alpha}$, $A\log n$, $A(\alpha-1)^{-1}$, according as $0<\alpha<1$, $\alpha=1$, $1<\alpha<\infty$, where in each case A depends only on α and the sequence $P_n(x)$ but not on n. The author twice uses an inequality due to Zygmund.

R. M. Redheffer (Cambridge, Mass.).

Loo, Ching-Tsün. On the uniform Cesàro summability of certain special trigonometrical series. Amer. J. Math. 72, 129–134 (1950).

Let S represent either the series $\frac{1}{2}c_0+\sum_{k=1}^\infty c_k\cos kx$ or the series $\sum_{k=1}^\infty c_k\sin kx$. The author proves that, for any non-negative integer k, if (*) $\Delta^{k+1}c_j\geqq 0$ and $c_j\to 0$, then the kth derived series of S is, for any $\epsilon>0$, uniformly summable (C, k) in $(\epsilon, \pi-\epsilon)$. The conclusion fails to hold if (*) is replaced by $\Delta^k c_j\geqq 0$. *P. Civin* (Eugene, Ore.).

Zamansky, Marc. Sur les séries trigonométriques. C. R. Acad. Sci. Paris 230, 44–46 (1950).

Let $f(x)$ be a continuous function of period 2π, $S_n(x)$ and $\sigma_n(x)$ the partial sums and the $(C, 1)$ means of the Fourier series of f, and $E_n[f]$ the best approximation of f by trigonometric polynomials of order not exceeding n. It is well known that the conditions $E_n[f]=O(1/n)$ and $E_n[f]=o(1/n)$ are respectively equivalent to $\Delta_2(f, x, t)=O(t)$ and $\Delta_2(f, x, t)=o(t)$, uniformly in x, where $\Delta_2(f, x, t)=f(x+t)+f(x-t)-2f(x)$ [see Zygmund, Duke Math. J. 12, 47–76 (1945); these Rev. 7, 60]. Let $\tilde{f}$ denote the function conjugate to f. The author states without proof that, if $E_n[f]=O(1/n)$, then each of the expressions

$$\int_0^h t^{-1}[f(x+t)-f(x)]dt,\quad \tilde{f}(x)-\tilde{\sigma}_n(x)\quad (n=[1/h]),$$

$$-h\pi^{-1}\int_h^\pi t^{-1}\{\tilde{f}(x+t)+\tilde{f}(x-t)-2\tilde{f}(x)\}dt$$

equals $f(x+h)-f(x)+O(h)$. If $E_n[f]=o(1/n)$, the error term is $o(h)$. [The result that $\tilde{S}_n(x)-\tilde{f}(x)=O(n^{-r})$ if $S_n(x)-f(x)=O(n^{-r})$, $r>0$, stated by the author, is not new; see Salem and Zygmund, Trans. Amer. Math. Soc. 59, 14–22 (1946); these Rev. 7, 435.] Finally, if the $S_n(x)$ are the partial sums of a trigonometric series T, and if the condition $S_n'(x)=o(n)$ is satisfied uniformly, then the series T integrated termwise converges absolutely and uniformly to a function $F(x)$ such that $h^{-1}[F(x+h)-F(x)]-S_N(x)\to 0$ $(N=[1/h])$. *A. Zygmund* (Chicago, Ill.).

Doronin, G. Ya. Some inequalities for approximation by trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 69, 487–490 (1949). (Russian)

Let $W^{(r)}$ be the class of functions of period 2π and having an rth derivative satisfying the inequality $|f^{(r)}(x)|\leqq 1$. Kolmogoroff showed [Ann. of Math. (2) 36, 521–526 (1935)] that the partial sums $S_n(x, f)$ of the Fourier series of f satisfy the relation

$$\sup_{f\epsilon W^{(r)}}|f(x)-S_n(x, f)|=4\pi^{-2}n^{-r}\log n+O(n^{-r}).$$

The function f for which the upper bound on the left is attained depends on n. The author here gives a sketch of the proof of the result that there is an $f_*\epsilon W^r$ independent of n such that

$$|f_*(x)-S_n(x, f_*)|>4\pi^{-2}(1-\epsilon_n)n^{-r}\log n$$

for infinitely many n, ϵ_n being a sequence tending to 0. Corresponding results are obtained for functions $f\epsilon$Lip α, $0<\alpha<1$, both for Fourier series and for Lagrange interpolation with equidistant abscissas. It is also observed that the phenomenon just described disappears if we consider approximation in the metric L or L^2. *A. Zygmund.*

Nikol'skiĭ, S. M. Fourier series of functions having a derivative of bounded variation. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 513–532 (1949). (Russian)

The topic of the paper is the degree of approximation, in the metric L^p, of a periodic function f by the partial sums $S_n(f)$ of the Fourier series of f. Let r be a positive number, not necessarily integral. A function $f(x)$ of period 2π is said to possess an rth derivative $f^{(r)}(x)$ in the sense of Weyl, equal to $\varphi(x)$, if $\varphi(x)$ is of period 2π, integrable L, satisfying

$\int_0^{2\pi}\varphi dx=0$ and such that

$$f(x)=\tfrac{1}{2}a_0+\pi^{-1}\int_0^{2\pi}D_0^{(r)}(t-x)\varphi(t)dt,$$

where $\frac{1}{2}a_0$ is the constant term of the Fourier series of f, and

$$D_0^{(r)}(t)=\sum_{k=1}^{\infty}k^{-r}\cos(kt+\tfrac{1}{2}\pi r).$$

If r is an integer, $\varphi(t)$ equals almost everywhere the ordinary derivative $f^{(r)}(t)$. We also set $f^{(0)}(x)=f(x)$. By $\omega(\delta,f)$ and $V(f)$ we shall denote respectively the modulus of continuity of f and the total variation of f over $0\leqq x\leqq 2\pi$. In all the results below it is assumed that f has an rth derivative $f^{(r)}$ of bounded variation. By A we shall denote constants independent of f and n. Finally, $\|f\|_p=(\int_0^{2\pi}|f|^p dx)^{1/p}$. The following results are obtained:

(1) $\|f-S_n(f)\|_p\leqq A\,V^{1/p}(f^{(r)})n^{-r-1/p}\omega(n^{-1},f^{(r)})^{1/p'}$ $(1<p\leqq+\infty,\ p'=p/(p-1))$;

(2) $|f(x)-S_n(x,f)|\leqq An^{-r}V(f^{(r)})$; (3) if $f^{(r)}$ is continuous, then $f(x)-S_n(x,f)=o(n^{-r})$; (4) let $x_1, x_2, \cdots$ be the points of discontinuity of $f^{(r)}$ in $0\leqq x<2\pi$, and let σ_k be the jump of $f^{(r)}$ at the point x_k; then

$$\|f-S_n(f)\|_p\simeq A_{r,p}(\sum|\sigma_k|^p)^{1/p}n^{-r-1/p},\qquad 1<p\leqq+\infty,$$

where the constant $A_{r,p}$ is given by a definite integral; $A_{0,\infty}=\frac{1}{2}$; (5) a necessary and sufficient condition for $f^{(r)}$ to be continuous is $\|f-S_n(f)\|_p=o(n^{-r-1/p})$, $1<p\leqq+\infty$; (6) $\|f-S_n(f)\|_1\leqq CV(f^{(r)})n^{-r-1}\log n$ (C, an absolute constant); (7) let $f_*^{(r)}x$ be such that $V(f_*^{(r)})=\inf V(f^{(r)}-\alpha)$, where the lower bound is taken with respect to all absolutely continuous functions $\alpha(x)$; then

(*) $\|f-S_n(f)\|_1\leqq 4\pi^{-2}V(f_*^{(r)})n^{-r-1}\log n+o(n^{-r-1}\log n)$;

(8) under the assumptions of (7), if f is a pure jump function, (*) is replaced by $\|f-S_n(f)\|_1\simeq 4\pi^{-2}n^{-r-1}\sum|\sigma_k|$.

A. Zygmund (Chicago, Ill.).

Mitra, S. C. On the sum of a series analogous to Fourier series. J. Indian Math. Soc. (N.S.) 13, 159–164 (1949).

If $f(t)$ is a periodic integrable function, then the standard sufficient conditions for the convergence (summability) of the Fourier series of $f(t)$ are shown to be sufficient for the convergence (summability) of the series

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\int_0^{2\pi}f(t)\cos\{n\pi\sin\tfrac{1}{2}(t-x)\}dt.$$

P. Civin (Eugene, Ore.).

Picone, Mauro. Vedute matematiche sull'analisi dei periodi. Rend. Sem. Mat. Fis. Milano 19 (1948), 17–30 (1949).

Picone, Mauro. Vedute matematiche sull'analisi dei periodi. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 5–6 (1949).

Expository lectures.

Bochner, S., and Chandrasekharan, K. Fourier series of L_2-functions. Duke Math. J. 16, 579–583 (1949).

Let $f(x_1, \cdots, x_k)$ be of period 2π in each x_j, of the class L^2, and let

$$S_\delta(R)=\sum_{\nu^2\leqq R^2}(1-\nu^2/R^2)^\delta a_{n_1,\cdots,n_k}e^{i(n_1x_1+\cdots+n_kx_k)},$$

where $\nu^2=n_1^2+\cdots+n_k^2$, be the typical means of the Fourier series of f. Let

$$f_0(t)\equiv f_0(x,t)=\tfrac{1}{2}\Gamma(k/2)\pi^{-k/2}\int_\sigma f(x_1+t\xi_1,\cdots,x_k+t\xi_k)d\sigma_\xi,$$

where σ is the sphere $\xi_1^2+\cdots+\xi_k^2=1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume element. Finally, for $p>0$, let

$$f_p(t)=f_p(x,t)=At^{-(2p+k-2)}\int_0^t(t^2-s^2)^{p-1}s^{k-1}f_0(s)ds,$$

where $A=2/\mathrm{B}(p,\frac{1}{2}k)$; then, if for a fixed point x we have

$$(*)\qquad\int_\lambda^{2\lambda}\{S^\delta(R)\}^2dR=o(\lambda),\qquad\lambda\to\infty,$$

for some $\delta>\frac{1}{2}(k-1)$, this implies for $p=\delta+\frac{1}{2}(3-k)$ the relation $\int_0^t\{f_p(u)\}^2du=o(t)$. The conclusion holds for $\delta=\frac{1}{2}(k-1)$, if (*) holds uniformly for $\frac{1}{2}(k-1)\leqq\delta<\delta_0$. *A. Zygmund.*

***Rodríguez Vidal, Rafael. Contribución al Estudio de las Sucesiones Casiperiódicas y sus Generalizaciones. [Contribution to the Study of Almost Periodic Sequences and their Generalizations].** Thesis, University of Barcelona, 1948. 73 pp.

A sequence (a_n), $n=\cdots,-1,0,1,\cdots$, is called almost periodic when there corresponds to every $\epsilon>0$ a relatively dense set of translation numbers, i.e., numbers τ satisfying $|a_{\nu+\tau}-a_\nu|\leqq\epsilon$ for all values of ν. Almost periodic sequences have been investigated earlier [A. Walther, Abh. Math. Sem. Hamburg. Univ. 6, 217–234 (1928); I. Seynsche, Rend. Circ. Math. Palermo 55, 395–421 (1931)]. The author adds little to these earlier results. He proves that an almost periodic sequence is normal, i.e., to a sequence $h_1, h_2, \cdots$ of integers corresponds a subsequence h_{p_ν} such that $(a_{n+h_{p_\nu}})$ converges uniformly towards a limit sequence when $\nu\to\infty$, and he mentions an example showing that a normal function is not always almost periodic. [This, however, is wrong and in his example the convergence of the sequence $(a_{n+h_{p_\nu}})$ is not uniform.] Relations are derived between the Fourier series of (a_n) and the Fourier series of partial sequences $(a_{p+n\lambda})$. In the second part of the paper the author studies asymptotically almost periodic sequences, i.e., sequences $(a_n)=(b_n+\epsilon_n)$, where (b_n) is almost periodic and $\lim_{n\to\pm\infty}\epsilon_n=0$. He further investigates almost periodic double sequences $(a_{m,n})$. These were generalized in the author's paper [Revista Mat. Hisp.-Amer. (4) 8, 239–242 (1948); these Rev. 11, 28]. An appendix contains a brief summary of some results of Jessen and Tornehave [Acta Math. 77, 137–279 (1945); these Rev. 7, 438]. *H. Tornehave* (Copenhagen).

***Bohr, Harald, and Jessen, Borge. Mean motions and almost periodic functions.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 75–84. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Expository paper. *R. H. Cameron.*

Kakehashi, Tetujiro. Stationary periodic distributions. J. Osaka Inst. Sci. Tech. Part I. 1, 21–25 (1949).

The author determines the stationary periodic distributions, found earlier by P. Lévy [Bull. Soc. Math. France 67, 1–41 (1939); these Rev. 1, 62], and discusses the convergence, as $n\to\infty$, of distributions obtained by convoluting a periodic distribution with itself n times.

P. Hartman (Baltimore, Md.).

Dugué, Daniel. Sur la structure des semi-groupes de variables aléatoires. C. R. Acad. Sci. Paris **230**, 50–52 (1950).

It is shown that there exist characteristic functions $\phi(z)$ such that $\phi(z)$ has no divisors (which are characteristic functions), whereas $\phi^n(z)$ has a divisor not of the form $\phi^m(z)$. Also, it is possible that for a $\psi(z)$ which is not a characteristic function two relatively prime powers $\psi^n(z)$ and $\psi^m(z)$ are characteristic functions. These statements are supported by examples of generating functions which are polynomials of fourth degree. *W. Feller* (Ithaca, N. Y.).

Dugué, Daniel. Sur certaines propriétés des lois indéfiniment divisibles. C. R. Acad. Sci. Paris **230**, 173–174 (1950).

An example is given of a characteristic function $\phi(z)$ such that $\phi^a(z)$ is characteristic for a certain nontrivial set of exponents which contains fractions but not all positive numbers. A few more theorems are stated which are consequences of the fact that characteristic functions of infinitely divisible laws cannot have zeros. *W. Feller.*

Cramér, Harald. On the factorization of certain probability distributions. Ark. Mat. **1**, 61–65 (1949).

With the exception of the normal distribution, each infinitely divisible distribution $F(x)$ is characterized by a nondecreasing function $M(x)$ which figures in the Kolmogorov-P. Lévy expression for the logarithm of the characteristic function $\phi(z)$. The author shows by a simple argument that $F(x)$ will have a factor $F_1(x)$ which is not itself infinitely divisible, if $M'(x)\geqq k>0$ almost everywhere in some interval $(0,c)$. Consider the monotonic function $M_1(x)$ whose derivative equals k in $(0,c)$ and 0 elsewhere; let $M_2'(x)=M_1'(x)$ except for $|x-c/2|<\epsilon$ where $M_2'(x)=(1+\epsilon)M_1'(x)$. Let $F_1(x)$ and $F_2(x)$ be the infinitely divisible distributions corresponding to the $M_j(x)$, and $\phi_j(z)$ their characteristic functions. Then $F(x)$ is divisible by $F_1(x)$. Now the author shows that $\phi_1(z)/\phi_2(z)$ is a characteristic function provided that ϵ is sufficiently small. Its logarithm is characterized by $M_1(x)-M_2(x)$ which is not monotonic, so that $F_1(x)\div F_2(x)$ is a not-infinitely divisible distribution. *W. Feller.*

Laplume, Jacques. Sur la réponse optimum à l'impulsion d'Heaviside d'un circuit à bande passante limitée. C. R. Acad. Sci. Paris **229**, 351–352 (1949).

The author considers the problem of a filter the output of which to a unit impulse reproduces the unit impulse in an optimum manner with a time lag. Optimum, following Wiener, is here taken as least square deviation. The problem leads easily to a second order differential equation with constant coefficients and an explicit solution. *N. Levinson* (Cambridge, Mass.).

***Schwartz, Laurent. Théorie des distributions et transformation de Fourier.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 1–8. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

This expository article gives various illustrations of the symmetry and convenient formalism in the theory of the Fourier transforms of the spherical distributions introduced by the author [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 7–24 (1948); these Rev. **10**, 36]. *I. E. Segal* (Chicago, Ill.).

Hirschman, I. I., Jr. On the behaviour of Fourier transforms at infinity and on quasi-analytic classes of functions. Amer. J. Math. **72**, 200–213 (1950).

In what follows let $f(x)$ belong to $L^2(-\infty,\infty)$ and let $\phi(t)$ be its Fourier transform. A general principle, due to N. Wiener, states that $f(x)$ and $\phi(t)$ cannot both tend to zero very rapidly for $|x|\to\infty$ unless both are nul-functions. [See Hardy, J. London Math. Soc. **8**, 227–231 (1933).] Theorems based on this principle were given by G. H. Hardy [loc. cit.], A. E. Ingham [J. London Math. Soc. **9**, 29–32 (1934)] and G. W. Morgan [ibid., 187–192 (1934)]. The author considers here a case where $\phi(t)$ tends to zero slowly while $f(x)$ tends to zero very rapidly. His main result is the following. Let (1) $\phi(t)\exp[|t|\theta(t)]\epsilon L^2(-\infty,\infty)$, $0\leqq\theta(t)\leqq M$; (2) $H(r)=\pi^{-1}\int_{-r}^{r}|t|\theta(t)[1+t^2]^{-1}dt$, $H(r)\to\infty$; (3) $f(x)=O(\exp[-L(x)])$ $(x\to+\infty)$, where $L(x)/x\uparrow\infty$. If $\limsup H[L(r)]r^{-1}>1$, then $f(x)=0$ almost everywhere. An example shows that the conditions of the theorem cannot be improved in a certain sense. Applications are given to infinitely differentiable functions belonging to a quasi-analytic class on the whole axis. *S. Agmon.*

Hirschman, I. I., Jr., and Widder, D. V. The inversion of a general class of convolution transforms. Trans. Amer. Math. Soc. **66**, 135–201 (1949).

Hirschman, I. I., Jr., and Widder, D. V. A representation theory for a general class of convolution transforms. Trans. Amer. Math. Soc. **67**, 69–97 (1949).

In these two papers the authors present an exhaustive study of the inversion and representation theory for transforms of the form $f(x)=\int_{-\infty}^{\infty}G(x-t)d\alpha(t)$. It is assumed that $G(x)$ is the inverse Laplace transform of the reciprocal of an entire function $E(s)$ of the form $E(s)=\prod(1-s/a_k)e^{s/a_k}$, a_k real, $\sum a_k^{-2}<\infty$. The main results have already been announced [Proc. Nat. Acad. Sci. U. S. A. **34**, 152–156 (1948); these Rev. **10**, 36], and the present papers are devoted to supplying the proofs. *H. Pollard.*

***Epstein, Benjamin. On a Certain Class of Transforms.** Abstract of a Thesis, University of Illinois, 1941. i+3 pp.

This is an abstract of the author's thesis which carries on work initiated by Bourgin and Duffin [Amer. J. Math. **59**, 489–505 (1937)] on the transform

$$\psi(s)=\int_A^\infty\int_0^\infty \nu^{s-1}e^{-\nu\lambda}\varphi(\nu)f(\lambda)d\nu d\lambda, \qquad A\geqq 0,$$

in which $\varphi(\nu)$ is assumed to be a fixed function. Throughout most of the discussion $\varphi(\nu)=(1-e^{-\nu})^{-1}$. The author is interested in relating asymptotic properties of $\psi(s)$ as $s\to1+$ with those of $f(\lambda)$ as $\lambda\to\infty$, and also in moment and closure problems associated with this transform. The abstract gives a brief summary of the problems discussed in the thesis, and states three theorems as samples of the results achieved. *A. Erdélyi* (Pasadena, Calif.).

Mitra, S. C. On certain self-reciprocal functions. Bull. Calcutta Math. Soc. **41**, 1–5 (1949).

A form of Poisson's summation formula given by Ramanujan [cf. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 62] is used to derive some self-reciprocal functions in the Hankel transform. Thus it is shown that the function

$$(\tfrac{1}{2}az)^\nu K_\nu(az)-(\tfrac{3}{2}az)^\nu K_\nu(3az)-(\tfrac{5}{2}az)^\nu K_\nu(5az)+\cdots$$

is $H_{2\nu-\frac{1}{2}}$ when $a=\frac{1}{2}\pi^{\frac{1}{2}}$, and similarly that the function

$F(az)=J_0(az)-J_0(3az)-J_0(5az)+\cdots$ is self-reciprocal in the sine transform for the same value of a.

M. C. Gray (Murray Hill, N. J.).

Mitra, S. C. On pairs of functions which are reciprocal in Fourier-sine transforms. Bull. Calcutta Math. Soc. **41**, 49–52 (1949).

This is a continuation of the paper reviewed above. Using the function $F(az)$ certain pairs of functions represented by series of Bessel and associated functions are shown to be sine transforms of each other. In particular, the function $F_2(ax)-F_2(3ax)-F_2(5ax)+\cdots$, where

$$F_2(x)=\int_0^\infty t^p(t^4+1)^{-\frac{1}{4}p-\frac{1}{4}}J_0(tx)dt,$$

is self-reciprocal in the sine transform. *M. C. Gray.*

Mitra, S. C. Corrections to my paper on 'Certain self-reciprocal functions.' Bull. Calcutta Math. Soc. **41**, 222 (1949).

Cf. the 2d preceding review.

Goldoni, Gino. Un teorema di passaggio al limite per la trasformata di Laplace. Atti Soc. Nat. Mat. Modena (6) **26**(79), 20–21 (1948).

Beweis der folgenden bekannten Bemerkung. Es gelte, gleichmässig für $0\leqq t<\infty$, $\lim_{n\to\infty}F_n(t)=F(t)$; $F_n(t)$ seien L-transformierbar, die entsprechende absolute Konvergenzabszisse sei β_n, $\limsup_{n\to\infty}\beta_n=\beta$ (endlich). Unter diesen Voraussetzungen ist auch $F(t)$ L-transformierbar.

W. Saxer (Zürich).

Ames, Dennis B. Certain inversion formulas for the Laplace transform. Proc. Amer. Math. Soc. **1**, 99–106 (1950).

An inversion formula for the Laplace transform is found for the case of a periodic determining function. The values of the generating function are required only at the points $\{c+in/k\}_{n=-\infty}^{\infty}$, where $c>0$ and $2k$ is the period of the determining function. When the length of the period increases without limit an inversion formula for an unrestricted determining function results which is essentially the complex inversion formula. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Bose, S. K. On Laplace transform of two variables. Bull. Calcutta Math. Soc. **41**, 173–178 (1949).

This paper contains seven results which are the straightforward analogues of the corresponding results in one-dimensional Laplace transform theory. *A. Erdélyi.*

Edrei, Albert. Sur des formules d'inversion pour les transformées de Stieltjès et certains théorèmes Taubériens. Ann. Sci. École Norm. Sup. (3) **66**, 395–408 (1949).

It is shown that if $f(z)=\int_0^\infty(z+t)^{-1}d\psi(t)$ and if

$$\psi(t)=\tfrac{1}{2}[\psi(t+)+\psi(t-)]$$

then

$$(2\pi)^{-1}\int_{-\pi+0}^{\pi-0}\Re[f(re^{i\theta})e^{i(p+1)\theta}]d\theta=(-1)^p r^{-p-1}\int_0^r t^p d\psi(t)$$

when $p\geqq 0$, together with a similar formula when $p<0$. These formulas are applied to obtain Tauberian theorems. The function $f(z)$ is analytic in the z-plane slit along the negative real axis. Let $g(z)$ be analytic there, let $g(x)\sim f(x)$ as $x\to+\infty$, and let $g(z)$ satisfy certain regularity conditions. If $\psi(t)$ is sufficiently restricted then we have that

$$(2\pi)^{-1}\int_{-\pi+\eta}^{\pi-\eta}\Re[g(re^{i\theta})e^{i(p+1)\theta}]d\theta$$

and

$$(-1)^p r^{-p-1}\int_0^r t^p d\psi(t)$$

differ by less than $\epsilon g(r)$ when $\eta=\eta(\epsilon)$ is sufficiently small and $r=r(\epsilon,\eta)$ is sufficiently large. *I. I. Hirschman, Jr.*

Castoldi, Luigi. Attorno a un teorema di calcolo operativo. Atti Accad. Ligure **5**, 218–220 (1949).

This is a generalisation of a result by D. Graffi [Mem. Accad. Sci. Ist. Bologna Cl. Sci. Fis. (10) **3**, 129–132 (1947); these Rev. **10**, 249]. If an operator (of any class) operating on $e^{at}t^k/k!$ produces $f(t)$ then the same operator operating on a function $V(t)$ of appropriate class results in

$$\left(\frac{d}{dt}-a\right)^{k+1}\int_0^t V(u)f(t-u)du.$$

[Reviewer's remark. Like Graffi's result, this is a simple example of the product theorem of Laplace transform theory.] *A. Erdélyi* (Pasadena, Calif.).

Ivanov, A. V. Generalization of the formula for the operational representation of the product of two functions. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 663–664 (1949). (Russian)

The following theorem is proved. If $f(p)=L\{F(t)\}$, and $g(w,p)=L\{e^{wa(t)}G(t)\}$, where $L\{F\}$ is the Laplace transform of F, then

$$L\{F[a(t)]\cdot G(t)\}=(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}f(w)g(w,p)dw$$

with the real number σ suitably chosen. The author states that the particular case of this theorem with $a(t)=t$ was given by G. A. Grünberg [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 141–143 (1943); these Rev. **6**, 127]. This particular case of the theorem, however, was given earlier by Bourgin and Duffin [Amer. J. Math. **59**, 489–505 (1937)].

H. P. Thielman (Ames, Iowa).

Delerue, Paul. Calcul symbolique à n variables et équations intégrales à n variables. C. R. Acad. Sci. Paris **229**, 916–919 (1949).

Straightforward generalisations of some results of the Heaviside calculus to n variables. No conditions of validity or proofs. *A. Erdélyi* (Pasadena, Calif.).

Delerue, Paul. Note sur une formule opératoire nouvelle en calcul symbolique. C. R. Acad. Sci. Paris **229**, 1197–1199 (1949).

Let $f(p)$ be the operational image of a known function. The author interprets $f(p^{-\nu})$ and $p^{-\nu}f(p^{-\nu})$. His result is a special case of a well-known general formula. Application to certain integral equations. The work is purely formal.

A. Erdélyi (Pasadena, Calif.).

Lebedev, N. N. The analogue of Parseval's theorem for a certain integral transform. Doklady Akad. Nauk SSSR (N.S.) **68**, 653–656 (1949). (Russian)

In connection with the integral transform formulae

$$G(y)=\int_0^\infty g(x)p(x,y)dx,\quad g(x)=\int_0^\infty G(y)p(x,y)dy$$

with $p(x, y) = \pi^{-1}(2y \sinh \pi y)^{\frac{1}{2}} x^{-\frac{1}{2}} K_{iy}(x)$ [N. N. Lebedev, same Doklady (N.S.) 65, 621–624 (1949); these Rev. 10, 604], the Parseval formula that the integrals of the squares of g and G are equal is proved under the assumptions (1) $g(x)x^{-\frac{1}{2}} \epsilon L(0, \infty)$, (2) $g(x) \epsilon L^2(0, \infty)$. [Reviewer's note: Condition (1) seems superfluous if the transforms are understood to exist in a mean square sense.]

J. L. B. Cooper (London).

Olevskiĭ, M. N. On the representation of an arbitrary function in the form of an integral with a kernel containing a hypergeometric function. Doklady Akad. Nauk SSSR (N.S.) 69, 11–14 (1949). (Russian)

Let $\bar{F}(a, b, c, x)$ denote the principal branch of the hypergeometric function [as defined, for example, in Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, p. 289]. The following theorem is proved. If (1) $\alpha+1>\gamma>\alpha>0$, $\gamma \geqq \frac{1}{2}$; (2) $x^\gamma e^{\pi x} f(x) \epsilon L(0, \infty)$; (3) $f(x)$ obeys Dirichlet's conditions on any finite segment of $(0, \infty)$, then

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \int_0^\infty \bar{F}(\alpha+xi, \alpha-xi, \gamma, -\xi) S_2(\alpha, \gamma, \xi) \varphi(\xi) d\xi,$$

where

$$\varphi(\xi) = \int_0^\infty \bar{F}(\alpha+\beta i, \alpha-\beta i, \gamma, -\xi) S_1(\alpha, \gamma, \beta) f(\beta) d\beta,$$

$$S_1(\alpha, \gamma, x) = \frac{x \sinh 2\pi x}{\pi^2[\Gamma(\gamma)]^2} \Gamma(\alpha+xi)\Gamma(\alpha-xi)\Gamma(\gamma-\alpha+xi)\Gamma(\gamma-\alpha-xi),$$

$$S_2(\alpha, \gamma, x) = x^{\gamma-1}(1+x)^{2\alpha-\gamma}.$$

It is stated, without proof, that by more exact estimates of the integrals involved the conditions of the theorem can be weakened to: (1) $\gamma>\alpha>0$, $\gamma>\frac{1}{2}$; (2) $x^{\gamma-\frac{1}{2}} f(x) \epsilon L(0, \infty)$ and (3) $f(x)$ of bounded variation in the neighbourhood of x.

J. L. B. Cooper (London).

Fine, N. J. On the Walsh functions. Trans. Amer. Math. Soc. 65, 372–414 (1949).

The author studies the properties of Walsh-Fourier series [WFS]. The Walsh functions are interpreted as group characters of the group of all infinite sequences of 0's and 1's, the group operation being addition mod 2 of the corresponding elements. This interpretation is found to be useful as a tool in studying WFS. A series of theorems deal with the order of magnitude of the coefficients c_n of the WFS of $f(x)$. It is proved that, if $f(x)$ satisfies a Lipschitz condition with exponent α $(0<\alpha\leqq 1)$, $c_n = O(n^{-\alpha})$; further if $f(x)$ is of bounded variation with total variation V, $|c_n| \leqq V/n$ $(n=1, 2, \cdots)$. Up to this point there is complete analogy with trigonometric Fourier series [TFS]. An essential difference (due to the fact that the Walsh functions are not continuous) is expressed by the fact that, for a nonconstant absolutely continuous $f(x)$, $c_n = o(n^{-1})$ cannot hold. As regards the Lebesgue constants L_n of the Walsh system, $L_n = O(\log n)$ holds as in the case of TFS but here $\lim_{n\to\infty} L_n/\log n$ does not exist; nevertheless the relation $n^{-1}\sum_{k=1}^{n} L_k = (\log n)/4 \log 2 + O(1)$ holds; the generating function $\sum_{n=1}^{\infty} L_n t^n$ is explicitly calculated. As regards convergence and summability, the WFS behave in some respect analogously to TFS. If $f(x)$ satisfies a Lipschitz condition of order $\alpha>0$, the WFS converges uniformly; if further $\alpha>\frac{1}{2}$ there is absolute convergence. If $f(x)$ is continuous, the first arithmetic means of its WFS converge uniformly to the function. An important difference arises at points of discontinuity of functions of bounded variation. It has been proved by Walsh [Amer. J. Math. 45, 5–24 (1923)] that if $f(x)$ is of bounded variation, and $f(x)$ is discontinuous at a point ξ which is not a dyadic rational point, the WFS of $f(x)$ does not converge at ξ. It follows by well-known Tauberian theorems that at such a point ξ the WFS of $f(x)$ cannot be summable either by some Cesàro method or by Abel's method. After discussing some sums related to those considered by M. Kac [J. London Math. Soc. 13, 131–134 (1938)] the author turns to theorems of uniqueness and localization. These problems he attacks by the method of the first formal integral of a Walsh series. It is proved that if a Walsh series converges to zero everywhere except on a denumerable set, then all coefficients vanish. [The same result was proved by A. A. Šneĭder; see the following review.] It is proved also that if a Walsh series converges everywhere and the sum of the series is integrable, then the series is the WFS of its sum.

A. Rényi (Budapest).

Šneĭder, A. A. On the uniqueness of expansions in Walsh functions. Mat. Sbornik N.S. 24(66), 279–300 (1949). (Russian)

Let $\varphi_n(x)$, $n=1, 2, \cdots$ $(0\leqq x\leqq 1)$ denote the Walsh functions [arranged following R. E. A. C. Paley, Proc. London Math. Soc. (2) 34, 241–264 (1932)]. A subset E of $(0, 1)$ is called a U-set (with respect to the Walsh system) if, from the supposition that (1) $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ converges to 0 for any x except for $x \epsilon E$, it follows that $c_n = 0$, $n=1, 2, \cdots$. A set E which is not a U-set is called an M-set. The following principal results are obtained. (I) Every measurable set E of positive measure is an M-set. (II) Every countable set is a U-set. (III) If E_1 and E_2 are U-sets and E_2 is closed, E_1+E_2 is also a U-set. (IV) If J_n $(n=1, 2, \cdots)$ are subintervals of $(0, 1)$ no two of which have common points, and if E_n is a U-set contained in J_n, it follows that $\sum_{n=1}^{\infty} E_n = E$ is also a U-set. (V) There exist perfect U-sets (of measure zero according to (I)). (VI) There exist M-sets of measure zero.

The proofs are based on the following localization theorem, proved in the paper. (VII) For any series (1) with $c_n \to 0$ and for any subinterval (a, b) of $(0, 1)$ with dyadic rational a and b, there exists a series $\sum_{n=1}^{\infty} d_n \varphi_n(x)$ which is uniformly equiconvergent with (1) in (a, b) and which converges uniformly to 0 outside $[a, b]$. The author proves (VII) by constructing a theory of formal multiplication of Walsh series, analogous to the theory of A. Rajchman [Math. Ann. 95, 389–408 (1925)]. From (VII) the following theorem is deduced. (VIII) Let us suppose that (1) converges to 0 in every point of a subinterval (c, d) of $(0, 1)$ except for points of a subset E of (c, d); it follows that E is either an M-set or empty. Theorems (III) and (IV) are straightforward consequences of (VIII); (V) is proved by means of the following general theorem. (IX) The set E is a U-set if there can be found a sequence $F_n(x)$ of functions having the following properties: (a) $F_n(x) = \sum_{k\leqq k_n} a_{nk}\varphi_k(x)$, (b) $\sum_{k\leqq k_n} |a_{nk}| \leqq C$ for $n=1, 2, 3, \cdots$, (c) $a_{n1}>a>0$ for $n=1, 2, \cdots$, (d) $\lim_{n\to\infty} a_{nk} = 0$ for $k=2, 3, \cdots$, (e) $F_n(x)=0$ for $x\epsilon E$ except for points belonging to a U-set $E_n \epsilon E$.

A. Rényi (Budapest).

Šneĭder, A. A. On the convergence of subsequences of the partial sums of Fourier series of Walsh functions. Doklady Akad. Nauk SSSR (N.S.) **70**, 969–971 (1950). (Russian)

Let $\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x), \cdots$ be the Walsh orthonormal system [Amer. J. Math. **45**, 5–24 (1923)] which [according to Paley, Proc. London Math. Soc. (2) **34**, 241–264 (1932)] can also be defined as follows. Let $r_0(x), r_1(x), \cdots$ be the Rademacher functions. Then $\varphi_N = r_{n_1}\cdot r_{n_2}\cdot\cdots$, if $N = 2^{n_1}+2^{n_2}+\cdots$ $(n_1 > n_2 > \cdots > 0)$. Any positive integer N can be represented, in general in many ways, in the form $N = \sum_0^m \theta_i 2^i$. The least possible number of θ_i distinct from zero here will be denoted by $\eta(N)$. The author states without proof a number of results concerning Walsh-Fourier series. The following are examples. (i) The Lebesgue constants L_N for the Walsh system satisfy the inequality $\eta(N)/3 \leqq L_N \leqq \eta(N)$ [see also the second preceding review]. (ii) Let $\omega(N)$ be an arbitrary function tending to $+\infty$ with N. If $S_N(x)$ are the partial sums of the Fourier series $\sum c_n\varphi_n \sim f$, then $S_N(x) = o(\max\{\eta(N), \omega(N)\})$ at every point of the Lebesgue set of f. (iii) The sequence $S_{N_p}(x)$ converges to f almost everywhere, provided $N_p = O(1)$. (iv) If positive numbers $W(n)$ are nondecreasing, and if $\sum c_n^2 W(n)$ is finite, then the subsequence $S_{N_p}(x)$ of the partial sums of $\sum c_n\varphi_n(x)$ converges, provided $\eta(N_p) \leqq W(N_p)$ for all p.

A. Zygmund (Chicago, Ill.).

Karlin, Samuel. Orthogonal properties of independent functions. Trans. Amer. Math. Soc. **66**, 44–64 (1949).

Let $x_n(t)$ denote a sequence of real measurable functions, defined on the interval $(0, 1)$, which are stochastically independent [for definition cf. M. Kac, Studia Math. **6**, 45–58 (1936)], and let us suppose that each $x_n(t)$ is bounded, further that $\int_0^1 x_n(t)dt = 0$ and $\int_0^1 x_n^2(t)dt = 1$, $n = 1, 2, \cdots$. The last two conditions imply that the functions $x_n(t)$ form an orthonormal system. Let us consider a series (1) $\sum_{n=1}^\infty a_n x_n(t)$ with real coefficients, and let us put $s_n(t) = \sum_{k=1}^n a_k x_k(t)$. The main results of the paper are as follows. The assumptions that $s_n(t)$ converges (a) almost everywhere [a.e.], (b) asymptotically, (c) strongly in L^p, (d) weakly in L^p to a function $f(t)\varepsilon L^p$ $(p \geqq 1)$ are equivalent. Moreover, the convergence in one of the four ways mentioned of an arbitrary subsequence of $s_n(t)$ implies the same for the entire sequence. If $s_n(t)$ converges a.e. to $f(t)\varepsilon L^p$, the series is unconditionally convergent a.e.; if further $f(t)$ is bounded a.e., (1) converges absolutely a.e. Let (b_{nk}) denote an arbitrary real Toeplitz matrix and let us put $\sigma_n(t) = \sum_{(k)} b_{nk}s_k(t)$. If $\sigma_n(t)$ converges (a) in the ordinary sense a.e., or (b) asymptotically, or (c) strongly in L^p or (d) weakly in L^p to a function $f(t)\varepsilon L^p$, the same holds also for $s_n(t)$. The additional condition $\liminf \int_0^1 |x_n(t)|^p dt > \Delta > 0$ imposed on the system $x_n(t)$ ensures that if (1) converges absolutely on a set E of positive measure, it converges absolutely a.e., further that if $\sigma_n(t)$ converges on a set E of positive measure, to a function $f(t)\varepsilon L^p$, $s_n(t)$ converges a.e. to $f(t)$ also.

Finally the notions of independent and lacunary orthogonal systems are compared with each other. It is proved that the above system $x_n(t)$ of independent functions is lacunary of order $p > 2$ [for definition cf. Kaczmarz and Steinhaus, Theorie der Orthogonalreihen, Warsawa-Lwów, 1935] if and only if $\int_0^1 |x_n(t)|^p dt$ are uniformly bounded. It is stated that the methods of the present paper yield new proofs of some known results for lacunary trigonometrical series, but not for general lacunary orthogonal systems.

A. Rényi (Budapest).

Sbrana, Francesco. Su una proprietà degli sviluppi in serie di funzioni ortogonali. Boll. Un. Mat. Ital. (3) **4**, 101–103 (1949).

Let $\{\varphi_n(x)\}$ be an orthonormal set of continuous functions on the interval $[a, b]$. There are given simple sufficient conditions in order that every function $f(x)$ having continuous first and second derivatives be developable in a convergent series $f(x) = \sum a_n\varphi_n(x)$.

B. de Sz. Nagy.

Wilkins, J. Ernest, Jr. The general term of the generalized Schlömilch series. Amer. J. Math. **72**, 187–190 (1950).

The author shows that an arbitrary interval may be replaced by a set of positive measure as the range of x for which the theorem of Watson holds [Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press; Macmillan, New York, 1944, p. 645; these Rev. **6**, 64]: the convergence to zero of the general term in the generalized Schlömilch series

$$\tfrac{1}{2}a_0/\Gamma(\nu+1) + \sum_{m=1}^{\infty} (\tfrac{1}{2}mx)^{-\nu}\{a_m J_\nu(mx) + b_m H_\nu(mx)\}$$

implies $a_m = o(m^{\nu+\frac{1}{2}})$, $b_m = o(m^{\nu+\frac{1}{2}})$, if $\nu < \frac{1}{2}$. The theorem remains true for $\nu = \frac{1}{2}$, and if $\nu > \frac{1}{2}$ then $a_m = o(m^{\nu+\frac{1}{2}})$, $b = o(m)$. These two cases were not treated by Watson.

R. G. Langebartel (Urbana, Ill.).

***Favard, J. Sur l'approximation des fonctions d'une variable réelle.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 97–110. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

A summary is given of recent progress in the theory of best approximation of functions of a real variable.

P. Civin (Eugene, Ore.).

Borg, Göran. On the completeness of some sets of functions. Acta Math. **81**, 265–283 (1949).

In the usual terminology, which is not the author's, the set $\{f_n(x)\} \subset L_p(a, b)$ is called complete in $L_p(a, b)$ if the equations $\int_a^b f_n(x)g(x)dx = 0$ imply $g(x) \sim 0$ whenever $g(x)\varepsilon L_r(a, b)$ $(1/r = 1 - 1/p)$. The author considers the equation (1) $y'' + \{\lambda - q(x)\}y = 0$, where $q(x)$, $q'(x)$ are continuous on $a \leqq x < b$ [these assumptions are scattered throughout the text]; b may be ∞, and $q(x)$ may have a singularity at $x = b$. Let $\psi_n(x)$ and $\psi_k^*(x)$ be the sets of the eigenfunctions of (1) with the boundary conditions (2) $y(a)\cos\alpha + y'(a)\sin\alpha = 0$, $y(x)\varepsilon L_2(a, b)$, and (3) (replace α by α^* in (2)), respectively, where $\alpha \not\equiv \alpha^* \pmod{\pi}$. In all cases where the condition $y(x)\varepsilon L_2(a, b)$ needs supplementing at $x = b$ (for example in the case where b is a regular point, which is a limit circle case) this is done. If the boundary value problems (1)+(2) and (1)+(3) both have point spectra with finite exponents of convergence (which is always true if b is a regular point), then the set (4) $\{\psi_n^2(x), \psi_k^{*2}(x)\}$ is complete in $L_1(a, b)$. The proof depends on the consideration of a boundary value problem which has the functions $\psi_n^2(x)$ and $\psi_k^{*2}(x)$ as its eigenfunctions. From now on, let b be finite, and a regular or regularly singular point of (1). Then the set (4) is complete in $L_2(a, b)$, and minimal after the exclusion of at most two functions. It is also proved that the set $\{\psi_n^2(x)\}$ is complete in $L_2(m, b)$, $m = \frac{1}{2}(a+b)$. After the exclusion of at most one function this set is minimal in $L_2(m-\epsilon, b)$ $(\epsilon > 0)$.

J. Korevaar (Lafayette, Ind.).

Hartman, Philip, and Wintner, Aurel. Derivative bases. Univ. Nac. Tucumán. Revista A. 7, 7–14 (1949).

I. Ibragimov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 389–390 (1946); Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 11, 75–100 (1947); these Rev. 8, 144, 509] proved that if $f(s)$, $s=t+iu$, is analytic in $|u|<\pi$ and of period 2π, then $\{f^{(n)}(s)\}$ $(n=0, 1, \cdots)$ is complete in $|s|\leqq\pi-\epsilon$ $(\epsilon>0)$ if no Fourier coefficient of $f(s)$ is zero. The authors prove the following result. Let $f(t)$ be analytic for real t and of period 2π: $f(t)\sim\sum c_m e^{imt}$, $c_m=O(e^{-\delta|m|})$, $m\rightarrow\pm\infty$, $\delta>0$ (hence $f(t+iu)$ analytic in $|u|<\delta$). Necessary and sufficient in order that $\{f^{(n)}(t)\}$ $(n=0, 1, \cdots)$ be complete in $C'[0, 2\pi]$ $(g(0)=g(2\pi)$ for $g(x)\epsilon C')$, or in $L^2(0, 2\pi)$, is that no c_m is zero. The condition of analyticity of $f(t)$ can be relaxed to quasi-analyticity in the sense that $c_m=O(e^{-\varphi(|m|)})$, $m\rightarrow\pm\infty$, $t\varphi'(t)\uparrow\infty$, $\int^\infty t^{-2}\varphi(t)dt=\infty$, but an example shows that the mere existence of derivatives of $f(t)$ of every order together with $c_m\neq0$ is not sufficient for completeness. In the proof the following lemma is used: the set $f_n(t)=\sum a_{nm}e^{imt}$, $(n=0, 1, \cdots)$, $\sum_m|a_{nm}|<\infty$, is complete in $C'[0, 2\pi]$ if the only solution of $\sum a_{nm}x_m=0$ with $x_m=O(1)$ is $x_m=0$ $(m=0, \pm1, \cdots)$. This follows from F. Riesz's theorem [compare A. Wintner, Quart. J. Math., Oxford Ser. 18, 209–214 (1947); these Rev. 9, 346]. *J. Korevaar* (Lafayette, Ind.).

Polynomials, Polynomial Approximations

Kössler, M. Some properties of trigonometric and algebraic polynomials. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1948, no. 15, 6 pp. (1949). (English. Czech summary)

Conditions necessary and sufficient in order that a trigonometric polynomial should be nonnegative (or that a corresponding algebraic polynomial should have a nonnegative real part in the unit circle) are established using its representation as a product. The results are applied to the problem of algebraic polynomials bounded in the unit circle. *O. Todd-Taussky* (Washington, D. C.).

Gavrilov, L. I. On the continuation of polynomials. Uspehi Matem. Nauk (N.S.) 4, no. 3(31), 181–182 (1949). (Russian)

The author states that, if $f(z)=1+a_1z+\cdots+a_nz^n$ is given and if C is a piecewise smooth closed Jordan curve enclosing the origin, then it is always possible to find a second polynomial of the form $f_1(z)=f(z)+a_{n+1}z^{n+1}+\cdots+a_mz^m$ such that all the roots of $f_1(z)$ lie on C. No indication of the method of proof is given. *A. W. Goodman.*

Parodi, Maurice. Sur une propriété d'une équation algébrique; application à l'étude des oscillations dans les réseaux électriques. C. R. Acad. Sci. Paris 229, 1190–1192 (1949).

Conditions necessary and sufficient are determined for the following determinantal equation in z to have its roots inside or outside the unit circle:

$$|a_{jk}^{(0)}z^m+a_{jk}^{(1)}z^{m-1}+\cdots+a_{jk}^{(m)}|=0$$

[for $m=2$ see also the same C. R. 228, 1400–1402 (1949); these Rev. 10, 671]. *O. Todd-Taussky.*

Montel, Paul. Sur les zéros des polynomes à coefficients réels associés à un polynome. C. R. Acad. Sci. Paris 229, 501–502 (1949).

The author states without proof the following theorem. Let $f(z)$, a polynomial of degree n with complex coefficients, be written in the form $f(z)=P(z)+iQ(z)$, where $P(z)$ and $Q(z)$ are real polynomials. Let p and q $(q\leqq p)$ denote the number of zeros of $f(z)$ in the upper and lower half planes, respectively. Then for every real m the polynomial $F(z)=P(z)+mQ(z)$ has at least $p-q$ real distinct zeros. Conversely, if for every real m the polynomial $F(z)$ has at least $n-2r$ and at most $n-2s$ real distinct zeros, then P/Q may be written in the form

$$P(z)/Q(z)=A_0+\sum_{j=1}^{u}A_j(z-a_j)^{-1}-\sum_{k=1}^{v}B_k(z-b_k)^{-1}+R(z),$$

where for $j=0, 1, \cdots, u=n-r-s$ and $k=1, 2, \cdots, v=r-s$, $0<A_j$, $0<B_k$ and the a_j and b_k are real, and where $R(z)$ is a real rational function with only nonreal poles. In the cases that $p=0$ or $q=0$ or that $r=s=0$, this theorem reduces to one due to Fujiwara [Tôhoku Math. J. 9, 102–108 (1916)]. *M. Marden* (Milwaukee, Wis.).

Varopoulos, Th. Sur le module des zeros des polynômes. Prakt. Akad. Athēnōn 17, 218–221 (1942). (Greek. French summary)

Let $\{c_n\}$ be a fixed infinite sequence of positive real numbers, P_n the set of all polynomials $x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$ with $|a_i|<c_i$, and $P=\bigcup_nP_n$. Let M be the totality of all roots of elements of P; the author is interested in conditions that M be a bounded set. The following results are stated. (1) A necessary and sufficient condition that M be bounded is that $\sum_1^\infty c_ix^i$ converge in some circle with positive radius; (2) if $r=\sup|z|$ for $z\epsilon M$ then $\sum_1^\infty c_ix^i$ converges in $|z|<1/r$; (3) if $\bar{r}$ is the positive real root of $c_1x+c_2x^2+\cdots+c_nx^n-1$ then the roots of the elements of P_n are all in $|z|<1/\bar{r}$. *J. Dugundji* (Los Angeles, Calif.).

Marković, Dragoljub. Sur la limite supérieure des modules des racines d'une équation algébrique. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. 6, 91–97 (1939).

The author derives a number of bounds for the zeros of a given polynomial $P(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}+z^n$ by use of an elementary inequality involving arbitrary real quantities r_k and arbitrary positive quantities b_k and c_k, namely $\min(r_k/b_k)\leqq(\sum_1^nr_kc_k)/(\sum_1^nb_kc_k)\leqq\max(r_k/b_k)$. Among the theorems so established are the following. (I) Let $Q(r)=\sum_0^{n-1}c_kr^k$ and $M=\max(|a_k|/c_k)$, $k=0, 1, \cdots, n-1$. Then all the zeros of $P(z)$ lie in the circle $|z|\leqq R$, where R is the positive root of the equation $r^n=MQ(r)$. (II) Let $S(r)=\sum_1^\infty c_kr^{-k}$ and $M=\max(|a_{n-k}|/c_k)$, $k=0, 1, \cdots, n-1$. Let the equation $MS(r)=1$ have R as its largest positive root for which $S(r)$ converges. Then all the zeros of $P(z)$ lie in $|z|\leqq R$. (III) Let $\delta_p=\sum_{k=1}^nc_k^{1-p}|a_{n-k-1}|^p$ and $S(r^q)=\sum_1^\infty c_kr^{-kq}$ where $p>0$, $q>0$ and $(1/p)+(1/q)=1$. Let the equation $S(r^q)\delta_p^{1/(p-1)}=1$ have R as its largest positive root for which $S(r)$ converges. Then all the zeros of $P(z)$ lie in the circle $|z|\leqq R$. These three theorems provide generalizations of some well-known bounds on the zeros of polynomials. Theorem (III) may be generalized further by use of the Hölder inequality. *M. Marden.*

Tsuji, Masatsugu. Algebraic equation, whose roots lie in a unit circle or in a half-plane. Proc. Japan Acad. 21 (1945), 313–320 (1949).

Let $f(z)=a_0+a_1z+\cdots+a_nz^n$, $\bar{f}(z)=\bar{a}_0+\bar{a}_1z+\cdots+\bar{a}_nz^n$ and $f^*(z)=\bar{a}_n+\bar{a}_{n-1}z+\cdots+\bar{a}_0z^n$ with $a_k=0$ for $k<0$ or

$k>n$. For j and $k=1, 2, \cdots, n$, let A, $\bar{A}'$, B and $\bar{B}'$ be the triangular matrices (c_{jk}) with $c_{jk}=a_{j-k}$, $\bar{a}_{k-j}$, $\bar{a}_{n+k-j}$ and a_{n+j-k}, respectively. Let the matrix $(h_{jk})=\bar{B}'B-\bar{A}'A$, noting that $h_{kj}=\bar{h}_{jk}$, and let the Hermitian form $H=\sum_0^{n-1}h_{jk}x_j\bar{x}_k$. It was shown by Cohn [Math. Z. **14**, 110–148 (1922)] that if H is of rank n and if it is reduced to the form $H=|y_1|^2+|y_2|^2+\cdots+|y_p|^2-|y_{p+1}|^2-|y_{p+2}|^2-\cdots-|y_n|^2$, then $f(z)$ has p zeros inside the unit circle $|z|=1$ and $n-p$ zeros outside $|z|=1$. In the present paper this theorem is proved by the study of the Hermitian form $D=\sum_0^{n-1}d_{k-j}x_j\bar{x}_k$, where the d_j are defined by the expansion

$$[f(z)-f^*(z)]/[f(z)+f^*(z)]=\sum_0^\infty d_jz^j$$

and the equation $d_{-k}=\bar{d}_k$. Also a theorem, similar to one due to Fujiwara [Math. Z. **24**, 161–169 (1925)] is proved for the number of zeros of $f(z)$ in the lower and upper half-planes $\Im(z)<0$ and $\Im(z)>0$; a quadratic form $E=\sum_0^{n-1}e_{j+k}x_jx_k$ is used with the e_j defined by the expansion

$$i[\bar{f}(z)-f(z)]/[\bar{f}(z)+f(z)]=\sum_0^\infty e_jz^{-j-1}.$$

[Reviewer's note: How to determine the number of zeros in a half-plane or circle without the use of quadratic or Hermitian forms is shown in the reviewer's book, The Geometry of the Zeros of a Polynomial in a Complex Variable, American Mathematical Society, New York, 1949, chapters 9 and 10; these Rev. 11, 101.] *M. Marden.*

v. Sz. Nagy, Gyula. Über Wertverteilung gebrochener rationaler Funktionen. Comment. Math. Helv. **23**, 288–293 (1949).

Let

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n$$

and

$$g(z)=z^n+b_1z^{n-1}+\cdots+b_n$$

be relatively prime polynomials and let $F(z)=f(z)/g(z)$. By a Z-point the author means a point z where $F(z)=Z$. Concerning these Z-points, he proves the following theorem. Let $f(\alpha)=g(\beta)=0$ and $C=f(\beta)/g(\alpha)$; if $t=|Z/C|^{1/n}<1$, then either at least one Z-point of $F(z)$ lies in the circle K: $|z-\alpha|<t|Z-\beta|$ or all the Z-points lie on K. The proof consists in studying the function

$$\varphi(z)=f(z)-Zg(z)=(a_p-Zb_p)\prod_{k=1}^{n-p}(z-z_k),$$

where a_p-Zb_p is the first nonzero quantity (a_k-Zb_k), $k=0, 1, \cdots, n$, and noting the relation

$$t^n=|-Z[g(\alpha)/f(\beta)]|=|\varphi(\alpha)/\varphi(\beta)|=\prod_{k=1}^{n-p}|\alpha-z_k|/|\beta-z_k|.$$

If every term of the latter product were to exceed t, the contradiction $t^n>t^{n-p}$ would follow. If, instead of the above, W is chosen so that $0<\omega=|\arg(-W/C)|\leqq\pi$, and, if S denotes the lens-region comprising all points from which the segment $\alpha\beta$ subtends an angle of at least ω/n, then the consideration of $\arg[\varphi(\alpha)/\varphi(\beta)]$ leads to the conclusion that either at least one W-point lies in S or all W-points lie on the boundary of S. If B is a convex region which contains all the zeros and poles of $F(z)$, the Z-points lie in the envelope of the circle K, as α and β vary independently over B, and the W-points lie in the envelope of the lens-region S. *M. Marden* (Milwaukee, Wis.).

v. Sz. Nagy, Gyula (Julius). Zur Nullstellenverteilung von Extremalpolynomen. Duke Math. J. **16**, 575–577 (1949).

A polynomial $A(z)=z^n+a_1z^{n-1}+\cdots+a_n$ is defined as the extremal polynomial of a given point set P if, for every polynomial $B(z)=z^n+b_1z^{n-1}+\cdots+b_n\not\equiv A(z)$, we have $|B(z)|<|A(z)|$ on P with $|B(z)|=|A(z)|$ only where $A(z)=0$. The distribution of the zeros of $A(z)$ has been studied by L. Fejér [Math. Ann. **85**, 41–48 (1922)], M. Fekete and J. L. von Neumann [Jber. Deutsch. Math. Verein. **31**, 125–138 (1922)] and the author [Acta Litt. Sci. Szeged **6**, 49–58 (1932)]. In generalizing the latter paper, the author proves the following. No equilateral hyperbola, which passes through a pair of zeros of $A(z)$ and has its center at the midpoint of the pair, can separate the points of P. As a supplement of the Fekete-Neumann result, he then deduces that, if P is symmetric in the real axis and if $A(z)$ is a real polynomial, not more than one zero of $A(z)$ can lie on the real axis in an interval exterior both to P and to the Jensen circles of P. A further deduction is that, if $P=P_1+P_2$, no multiple zero of $A(z)$ may lie at a point from which both P_1 and P_2 separately subtend an angle less than $\pi/2$. *M. Marden* (Milwaukee, Wis.).

Neĭmark, Yu. I. *D*-decomposition of the space of quasi-polynomials. (On the stability of linearized distributive systems). Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 349–380 (1949). (Russian)

A sum $Q(z)=\sum_{k,s=0}^{m,n}a_{ks}z^ke^{\tau_sz}$ with complex coefficients a_{ks} and real exponents τ_s is called a quasipolynomial. Consider a family of quasipolynomials depending on some parameters; by a D-decomposition of this family is meant a decomposition of the parameter space into the sets $D(0)+D(1)+\cdots+D(\infty)$, where $D(l)$ consists of the parameter points for which $Q(z)$ has precisely l zeros with positive real part. The author is interested in the number of roots of $Q(z)=0$ in the right half-plane, a question of importance in the study of the stability of certain dynamical systems. For this purpose he investigates the D-decomposition when the parameter space consists of the coefficients and exponents of Q as well as when the exponents are fixed and the parameter space is the space of coefficients. A more detailed study is made of the D-decomposition of the complex w-plane and of the real (u, v)-plane when $Q(z)=A(z)+wB(z)$ and $Q(z)=uA(z)+vB(z)+C(z)$, respectively, A, B and C being given quasipolynomials. The method of study centers on the consideration of how a continuously varying parameter point can pass from one $D(l)$ set to another. It follows earlier work by the author [for references see the 2d review cited below] and L. Pontrjagin [Bull. Acad. Sci. URSS, Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 115–134 (1942); these Rev. **4**, 214]. Some of the results of this paper were published earlier [Doklady Akad. Nauk SSSR (N.S.) **60**, 1503–1506 (1948); these Rev. **10**, 37]. Considerations of space make it impossible to quote here any results beyond those given in the review referred to. [The paper contains various inaccuracies but these do not seem to be of great importance.]

A. Dvoretzky (Princeton, N. J.).

Vazsonyi, A. A generalization of Nyquist's stability criteria. J. Appl. Phys. **20**, 863–867 (1949).

A type of diagram is described which is of similar nature to a Nyquist plot. If a conjectured lower limit to the values of the damping ratios is selected, and the new type of curve is drawn for this value, it is possible to decide whether or

not the conjecture was correct from the relation of the new curve to the origin. *P. Franklin* (Cambridge, Mass.).

Mason, S. J. A comment on Dr. Vazsonyi's paper, "A generalization of Nyquist's stability criteria." J. Appl. Phys. 20, 867 (1949).

[Cf. the preceding review.] Presents a graphical method of estimating critical frequencies from a standard Nyquist plot of $F(j\omega)$. *P. Franklin* (Cambridge, Mass.).

Nikol'skiĭ, S. M. On the asymptotically best linear method of approximating differentiable functions by polynomials. Doklady Akad. Nauk SSSR (N.S.) **69**, 129–132 (1949). (Russian)

Let W^r be the class of functions $f(x)$ defined on the interval $-1\leqq x\leqq +1$, having an rth derivative there and satisfying the condition $|f^{(r)}(x)|\leqq 1$. Similarly, let W_*^r be the class of functions of period 2π, differentiable r times and such that $|f^{(r)}(x)|\leqq 1$. Let $E_n(f)$ be the best approximation of an $f\epsilon W^r$ by polynomials of degree n, and $E_n^*(f)$ the best approximation of an $f\epsilon W_*^r$ by trigonometric polynomials of order n. It was proved by S. Bernstein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 583–585 (1946); same Doklady (N.S.) **57**, 3–5 (1947); these Rev. **8**, 378; **9**, 179] that

$$(1)\qquad \sup_{f\epsilon W_*^r} E_n(f) \simeq \sup_{f\epsilon W^r} E_n(f), \qquad n\to\infty.$$

It had previously been proved by J. Favard [Bull. Sci. Math. (2) **61**, 209–224, 243–256 (1937)] and by Akhieser and Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **15**, 107–111 (1937)] that there exists a linear method of summability of Fourier series,

$$U_n(f,x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}\lambda_k^n(a_k\cos kx+b_k\sin kx)$$

which for the whole class W_*^r gives the same approximation as $E_n(f)$, that is,

$$(2)\qquad \sup_{f\epsilon W_*^r} E_n(f)=\sup_{f\epsilon W_*^r}|f(x)-U_n(x,f)|.$$

Using (1), the author constructs a linear method of approximation of functions $f\epsilon W^r$ by polynomials, which though it does not give the analogue of (2) (with asterisks omitted) gives asymptotic equality of both sides for $n\to\infty$. An explicit form of these polynomials is not simple.
A. Zygmund (Chicago, Ill.).

Salzer, Herbert E. Polynomials for best approximation over semi-infinite and infinite intervals. Math. Mag. **23**, 59–69 (1949).

Let $p_n(x)=x^n+\cdots$ be the normalized polynomial of degree n for which $\max e^{-x}|p_n(x)|$, $x\geqq 0$, is a minimum. For the extremum polynomial this maximum is assumed $n+1$ times with alternate signs, one of these maxima being assumed at $x=0$. This condition leads to certain equations from which $p_n(x)$ can be computed at least for $n=1$ and $n=2$. The corresponding problem for $e^{-x^2}|q_n(x)|$, $-\infty<x<\infty$, is also treated and $q_3(x)$ computed. *G. Szegö.*

Merli, Luigi. Alcune osservazioni sulla interpolazione delle funzioni discontinue. Boll. Un. Mat. Ital. (3) **4**, 140–146 (1949).

Au sujet de l'approximation de la fonction $y=\operatorname{sgn} x$ $(-1\leqq x\leqq 1,\ \operatorname{sgn} 0=0)$ par des polynomes interpolateurs d'Hermite, $y=H_n(x)$, de degré $8n+1$ ($\operatorname{sgn} x=\operatorname{sgn} H_n(x)$), formés à partir des polynomes de Tchebitchef de première espèce, l'auteur montre que, pour $x>0$ (ou $x<0$), le nombre des arches des courbes $y=H_n(x)$ qui s'aplatissent sur l'intervalle $0<y<1$ (ou $-1<y<0$) augmente indéfiniment avec n. Cette remarque est interessante car les phénomènes analogues à celui de Gibbs ne rentrent encore dans aucune théorie générale. *J. Favard* (Paris).

Kakehashi, T. Interpolating orthogonal polynomials and the convergence of interpolations. J. Osaka Inst. Sci. Tech. Part I. **1**, 5–11 (1949).

Let $x_0<x_1<\cdots<x_n$ be given abscissas, $\mu_n(x)$ the distribution function with equal jumps at x_i. The author forms the orthonormal polynomials $\phi_m^{(n)}(x)$ and the kernel $K_m^{(n)}(x,y)$, $m\leqq n$, associated with the distribution $d\mu_n(x)$. It is well known that for the Lagrange interpolating polynomial $S_n(x,f)$ of a function f we have the representation

$$S_n(x,f)=\int_{-\infty}^{\infty}f(y)K_n^{(n)}(x,y)d\mu_n(x).$$

Using this formula a bound for the error $f(x)-S_n(x,f)$ can be obtained involving the "best approximation" $E_n(f)$ and $\max|\phi_m^{(n)}(x)|$, $m\leqq n$, $a\leqq x\leqq b$. [It is not clear how a bound can be obtained by using merely a bound for $\phi_n^{(n)}(x)$ (see (2.4) on p. 7).] *G. Szegö* (Stanford University, Calif.).

Hahn, Wolfgang. Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören. Math. Nachr. **2**, 263–278 (1949).

A chain is defined as a sequence of polynomials $\varphi_n(x)$ satisfying a recurrence equation:

$$\varphi_n(x)=(a_nx+b_n)\varphi_{n-1}(x)-c_n\varphi_{n-2}(x),$$

with a_n, b_n, c_n independent of x. If $d\psi(x)$ is a weighting of the interval (A,B), the associated orthogonal polynomials $\varphi_n(x,d\psi(x))$ form a chain with $c_n>0$. Suppose that $d\psi$ depends on a second variable y, and let $p_1(x), p_2(x),\cdots$ be polynomials in x all of the same degree. For fixed m, the sequence $u_n(x,y)=\varphi_n(x,y,p_1(x),\cdots,p_{m-n}(x),d\psi(x))$ satisfies a three-term recurrence. The author seeks to make $u_n(x,y)$ a chain with respect to y. This leads to certain divisibility conditions on the moments $\int_A^Bx^id\psi(x)$. Special cases are then considered: (a) $p_i=x(x-\beta_i)$, $\beta_i\neq 0$. These lead to (finite) q-series previously considered by the author [same vol., 4–34 (1949); these Rev. **11**, 29]. The limiting case $q=1$ yields the Jacobi polynomials in x, the Krawtchouk polynomials in y. (b) $p_i=x^2$. This is a trivial case in which the two recurrence equations are identical. (c) $p_i=x$. These again lead to q-series which reduce for $q=1$ to the Laguerre and Charlier polynomials. (d) $p_{2i-1}=x$, $p_{2i}=x-1$. Here the moments must all vanish, so the proposed problem has no solution. The method used to show that the series for cases (a) and (c) actually are solutions leads also to q-series of a type not previously considered by the author.

The last section deals briefly with polynomials orthogonal on the unit circle with respect to a weight-function $f(z)$. Results are obtained which are specialized to a case considered by Szegö [S.-B. Preuss. Akad. Wiss. **1926**, 242–252]: $f(z)=\sum_{n=-\infty}^{\infty}q^{n^2/2}z^n$; the corresponding polynomials are

$$G_n(z)=\sum_{\nu=0}^{n}\left[{n\atop \nu}\right](-1)^\nu q^{-\nu/2}z^\nu,$$

where the $\left[{n\atop \nu}\right]$ are the Gauss polynomials in q. [Note. The $G_n(z)$ are of considerable independent interest. They have been evaluated simply for a few special values of z; further nontrivial evaluations would be highly useful.]
N. J. Fine (Philadelphia, Pa.).

Harmonic Functions, Potential Theory

Reade, Maxwell O. Harmonic polynomials. Duke Math. J. 16, 627–631 (1949).

Soit n entier fixe $\geqq 2$, φ un angle $(-\pi/n \leqq \varphi < \pi/n)$ et $\zeta_m = e^{i(\varphi+(2m-1)\pi/n)}$. S'appuyant sur des résultats antérieurs [Beckenbach et Reade, Bull. Amer. Math. Soc. 50, 336 (1944)] l'auteur montre que si $f(z)$ est une fonction reélle continue dans le domaine D: $|z|<1$, une condition nécessaire et suffisante pour que

$$\sum_{m=1}^{n} f(z+\alpha\zeta_m) f(z+\beta\zeta_m) = \sum_{m=1}^{n} [f(z+\gamma\zeta_m)]^2$$

où $\gamma = (\alpha\beta)^{\frac{1}{2}}$, pour tout système de nombres positifs α, β tels que les polygones de sommets $z+\alpha\zeta_m$ et $z+\beta\zeta_m$ soient dans D, est que f soit un polynôme harmonique de degré au plus égal à la partie entière de $(n/2-1)$, et d'une certaine forme précisée si on fixe φ. *M. Brelot* (Grenoble).

Verblunsky, S. Sur les fonctions préharmoniques. Bull. Sci. Math. (2) 73, 148–152 (1949).

Démonstration du théorème: soit f une fonction préharmonique sur le réseau des points (x, y) tels que x et y soient entiers et $-s \leqq x \leqq +s$, $-s \leqq y \leqq +s$ (s entier). Alors $|f(1, 0)-f(0, 0)| < 50s^{-1} \sup |f|$. L'auteur ne cherche pas la meilleure constante par laquelle on peut remplacer 50. *H. Cartan* (Paris).

Minakshisundaram, S. A generalization of Epstein zeta functions. With a supplementary note by Hermann Weyl. Canadian J. Math. 1, 320–327 (1949).

Given the Laplace equation $\sum_{i=1}^{k} \partial^2 u/\partial x_i^2 + \lambda u = 0$ in a Euclidean domain D with smooth boundary B, if $\{\omega_n(x)\}$ and $\{\lambda_n\}$ are eigenfunctions and eigenvalues corresponding to the classical boundary condition $u=0$ or $\partial u/\partial n = 0$, then the series $\sum_{n=1}^{\infty} \omega_n(x)\omega_n(y)\lambda_n^{-s}$ is an entire function of s for $x \neq y$, and it has a simple pole at $s=k/2$ for $x=y$ with a known residue. The proof employs a lemma to the effect that the Green's function of the associated heat equation satisfies an estimate $|G(x, y, t)| \leqq Ct^{-\frac{1}{2}k} \exp(-ly^2/4t)$ where ly is the distance of y from the boundary. The supplement gives an alternate proof of the lemma for the first boundary condition. *S. Bochner* (Princeton, N. J.).

Berker, Ratip. Sur certaines propriétés du rotationnel d'un champ vectoriel qui est nul sur la frontière de son domaine de définition. Bull. Sci. Math. (2) 73, 163–176 (1949).

This paper contains the proofs of results summarized in C. R. Acad. Sci. Paris 228, 1630–1632 (1949); these Rev. 11, 107. *N. Coburn* (Ann Arbor, Mich.).

Calderón, A. P. On the behaviour of harmonic functions at the boundary. Trans. Amer. Math. Soc. 68, 47–54 (1950).

Extension à l'espace de certaines généralisations du théorème de Fatou: Si $F(P) = F(x_1, \cdots, x_n)$ est une fonction harmonique pour $x_n > 0$ telle que, en chaque point Q d'un ensemble E de l'hyperplan $x_n = 0$, il existe une région limitée par un cone de sommet Q et un hyperplan $x_n =$ constante, dans laquelle F reste bornée, alors presque partout dans E, la fonction $F(P)$ a une limite lorsque P tend vers Q non tangentiellement à $x_n = 0$. Ce résultat s'étend aux fonctions analytiques de n variables complexes. *J. Lelong*.

Calderón, A. P. On a theorem of Marcinkiewicz and Zygmund. Trans. Amer. Math. Soc. 68, 55–61 (1950).

Extension à l'espace à $n>2$ dimensions d'un théorème démontré par Marcinkiewicz et Zygmund pour $n=2$ par des méthodes de représentation conforme [Duke Math. J. 4, 473–485 (1938)]: si $F(x_1, x_2, \cdots, x_n)$ est harmonique pour $x_n > 0$, et si pour chaque point Q d'un ensemble E de l'hyperplan $x_n = 0$, il existe une région Γ_Q limitée par un cone de sommet Q et un hyperplan $x_n = Cte$ dans laquelle F reste bornée, alors, excepté pour un ensemble de mesure nulle, l'intégrale $\int x_n^{-n+2} \operatorname{grad}^2 F d\omega$ étendue à une région quelconque limitée par un cone de sommet Q situé dans $x_n > 0$ et par un hyperplan $x_n =$ constante, est finie. *J. Lelong* (Lille).

Lorentz, G. G. Minimalfolgen und das Dirichletsche Prinzip. Math. Z. 51, 696–699 (1949).

Let G denote a bounded domain (connected open set (in Euclidean n-space, B its boundary, and $f(p)$ a continuous function on B. A function $v(P)$, $P \epsilon G+B$, is termed admissible if it satisfies the following conditions: (1) v is continuous on $G+B$, and $v=f$ on B; (2) the first partial derivatives of v exist in G, except perhaps on a subset whose $(p-1)$-dimensional Carathéodory measure is finite; (3) the Dirichlet integral $D(v)$ of v over G is finite. Assume now that there exist admissible functions for an assigned boundary value function f. Let d be the greatest lower bound of $D(v)$ for all admissible functions. A minimal sequence v_n is a sequence of admissible functions such that $D(v_n) \rightarrow d$. The following results are established. (a) If v_n is a minimal sequence, then v_n converges in G to a harmonic function u in the sense of means of order two. As a corollary, this harmonic function u is independent of the choice of the minimal sequence v_n. (b) The function u coincides with the Wiener solution of the boundary value problem $\Delta u = 0$ for the assigned boundary value function f. (c) The minimal sequence can be so chosen that v_n converges to u uniformly on every closed subset of G. These results complete earlier ones obtained by the author jointly with Kamke [same vol., 217–232 (1948); these Rev. 10, 116]. *T. Radó*.

Kakutani, Shizuo. Markoff process and the Dirichlet problem. Proc. Japan Acad. 21 (1945), 227–233 (1949).

Let D be a bounded plane domain, let C be a closed circular disc in D and let $x(z)$ be continuous on $\bar{D}$. Let $P(C)x$ be the function equal to $x(z)$ except in the interior of C, where it is the harmonic function determined by $x(z)$ on the perimeter of C. Then $P(C)$ can be represented as an integral operator acting on the functions $\{x(z)\}$ and the corresponding kernel can be identified as the transition probability function of a Markov process. In the classical sweeping out method a sequence of circles $C_1, C_2, \cdots$ is chosen in a certain way and then $P(C_n) \cdots P(C_1)x$ converges for all x, when $n \rightarrow \infty$, to the solution of the Dirichlet problem with the boundary values those of $x(z)$. It is stated that this is also true of $P(C_1) \cdots P(C_n)x$ and of $R^n x$, where Rx is the function equal at each z to the integral average of x in a certain circle with center at z. In terms of the kernels of these operators the statement is that the kernels corresponding to the above operator products converge in a weak sense to a kernel $P(z, E)$ which is the harmonic measure of the set E with respect to the point z. Since the kernel composition corresponding to an operator product is precisely the composition used in the theory of

Markov processes, the theorems obtained can be considered as theorems on certain Markov processes (which in the case of R have stationary transition probabilities).

J. L. Doob (Ithaca, N. Y.).

Amerio, Luigi. Sui problemi di Cauchy e di Dirichlet per l'equazione di Laplace in due variabili. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 57–70 (1949).

For a plane region bounded by a circle or an ellipse the author sets up the solution of the Cauchy problem for the Laplace equation, and indicates carefully the method of passing from this solution to the solution of the Dirichlet problem for this same region. This problem is briefly discussed for slightly more general plane regions, and the author again presents his method of solving the Fredholm equation arising in the classical treatment of the Dirichlet problem [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 393–425 (1943); these Rev. 9, 32]. *F. G. Dressel.*

Tautz, Georg. Zur Theorie der ersten Randwertaufgabe. Math. Nachr. 2, 279–303 (1949).

This paper contains an extension of the theory developed by the author in an earlier paper [Math. Ann. 118, 733–770 (1943); these Rev. 6, 3]; some of the methods used are related to those employed by Perron [Math. Z. 18, 42–54 (1923)]. A large part of this new formulation is carried through for a metric space Ω_0 of considerable generality. A system $\mathfrak{S}$ of spheres κ in Ω_0 is such that if $\kappa\epsilon\mathfrak{S}$ and $\kappa'\subset\kappa$ then $\kappa'\epsilon\mathfrak{S}$, and if $P\epsilon\Omega_0$ then P is the center of at least one κ. To each κ and each continuous function f defined in κ the author assigns a continuous function $L_{f,\kappa}(P)$ in accordance with a set of postulates. A function $u(P)$ is regular at P if it is continuous in the neighborhood of P and if for every $\kappa^*\subset\kappa$ we have $u(P)=L_{u,\kappa^*}(P)$. Let a bounded function $f(R)$ be defined on the boundary of an arbitrary bounded open set Ω. The boundary value problem studied considers the possibility of determining a function u, regular throughout Ω, which satisfies the boundary condition

$$\underline{f(R)}\leqq\underline{u(R)}\leqq\overline{u(R)}\leqq\overline{f(R)}.$$

(Here the bars below and above indicate inferior and superior limits at R.) Barriers are discussed and various conditions for the regularity of boundary points are given. In the latter part of the paper it is assumed that the space Ω_0 is Euclidean. Analogues are introduced for Green's function and for capacity. A result is given which corresponds to Wiener's condition for the regularity of a boundary point [J. Math. Physics 3, 24–51, 127–146 (1924)].

F. W. Perkins (Hanover, N. H.).

Kunugui, Kinjiro. Sur quelques points de la théorie du potentiel. I. Proc. Japan Acad. 21 (1945), 234–239 (1949).

L'auteur considère dans l'espace à 3 dimensions (en indiquant la possibilité d'extension à $n\geqq2$ dimensions) les potentiels de masses $\geqq0$ du type $\int\Phi(1/MP)d\mu_P$ où $\Phi(t)$ est croissante convexe de limite $+\infty$ pour $t\to+\infty$. Il démontre essentiellement le principe correspondant du maximum: si ce potentiel est majoré par K sur la frontière d'un domaine sans masses, il l'est aussi dans ce domaine. Il utilise un passage à la limite sur les potentiels "quasi-newtoniens," c'est-à-dire correspondant à un $\Phi(t)$ linéaire pour t assez grand. *M. Brelot* (Grenoble).

Myškis, A. D. On a criterion for subharmonicity. Mat. Sbornik N.S. 25(67), 315–320 (1949). (Russian)

L'auteur montre d'abord que si u est continuement différentiable dans un domain G, une condition nécessaire et suffisante de sousharmonicité est que l'intégrale-flux $\int(du/dn)ds$ vers l'intérieur soit $\leqq0$ pour toute surface assez regulière contenue dans G ainsi que son intérieur, propriété bien connue depuis l'introduction des fonctions sousharmoniques. L'extension en est faite aux fonctions analogues, comparées non plus aux fonctions prenant les mêmes valeurs frontière et harmoniques mais intégrales de

$$(1)\qquad \sum\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right)=0,\qquad a_{ik}=a_{ki},$$

$\sum a_{ik}x_ix_k$ étant une forme quadratique définie positive et les a_{ik} à dérivées secondes höldériennes. Le flux est remplacé par $\sum a_{ik}\cos(x_i,n)(\partial u/\partial x_k)ds$ et la condition de signe équivaut aussitôt à

$$\sum\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right)\geqq0,$$

ce dont on voit l'équivalence avec la définition (on utilise des résultats de Giraud sur la résolution du problème de Dirichlet pour l'équation (1) et l'existence de dérivées normales de la solution à la frontière). *M. Brelot.*

Bonsall, F. F. Note on a theorem of Hardy and Rogosinski. Quart. J. Math., Oxford Ser. 20, 254–256 (1949).

Elementary proof of a known theorem. If $f(x, y)$ is subharmonic in the strip $\alpha<x<\beta$, $y>\gamma$, then

$$\varphi(x)=\limsup_{y\to\infty}f(x, y)$$

is either $-\infty$ or convex for $\alpha<x<\beta$ [Hardy and Rogosinski, Proc. Roy. Soc. London. Ser. A. 185, 1–14 (1946); these Rev. 7, 448]. This theorem is a particular case of a result of M. Brelot and H. Cartan [Cartan, Bull. Soc. Math. France 73, 74–106 (1945); these Rev. 7, 447] completed by J. Deny and P. Lelong [ibid. 75, 89–112 (1947); these Rev. 9, 352]. *J. Lelong* (Lille).

Lelong-Ferrand, Jacqueline. Extension du théorème de Phragmen-Lindelöf-Heins aux fonctions sous-harmoniques dans un cône ou dans un cylindre. C. R. Acad. Sci. Paris 229, 411–413 (1949).

In this note, the author states various generalizations of results concerning the Phragmen-Lindelöf theorem due to Heins and to herself [for notation see the same vol., 340–341 (1949); these Rev. 11, 176]. For example, let D denote a cone with vertex O, frontier C, in Euclidean space E^p, $p\geqq2$. Let Σ_R denote the spherical cap cut by D on the sphere Σ with center O, radius R, let D_R denote the subdomain of D defined by $\overline{OM}<R$, and let $\omega_R(M)$ denote the harmonic measure of Σ_R in D_R. Now $\omega_R(M)$ is harmonic in D_R, so that it has an expansion of the form

$$\omega_R(M)=\sum_{i=0}^{\infty}[A_i(r/R)^{\alpha_i}+B_i(r/R)^{\beta_i}]\varphi_i(m),$$

where $\overline{OM}=r$, where A_i and B_i are constants and where m is the point in which $\overline{OM}$ intersects Σ_1. A typical result is the following. Let $u(M)$ be harmonic in D and satisfy (i) $\lim_{M\to P}u(M)\leqq0$, for all $P\epsilon C$, (ii) $\liminf_{r\to\infty}\mu(r)/r^{\alpha_0}=\lambda<\infty$, (where $\mu(r)=\sup u(M)$ for $\overline{OM}=r$); then $u(M)<\lambda A_0r^{\alpha_0}\varphi_0(m)$ holds throughout D. *M. O. Reade* (Ann Arbor, Mich.).

Differential Equations

Rubbert, F. K. Über Schwingungen mit kombinierter Dämpfung. Ing.-Arch. **17**, 165–166 (1949).

Detailed solution of the equation for damped vibration with a constant friction force opposed to the velocity.

P. Franklin (Cambridge, Mass.).

Viguier, G. Les chaînes de Darboux et l'équation de Fourier. Experientia **5**, 439 (1949).

A chain of Riccati equations, the solution of each of which follows from that of the preceding, is derived. From these a series of solutions of the heat equation can be found. A geometric interpretation is also given. *P. Franklin.*

Vakselj, Anton. La fonction de monodromie de l'équation différentielle linéaire. Akad. Ljubljani. Mat.-Prirodoslov. Razred. Mat. Odsek. Razprave **3**, 7–18 (1947). (Slovenian. French summary)

Let y_1, y_2 be two independent solutions of the equation $y''-py=0$ with Wronskian $W[y_1(x), y_2(x)]=W(x)$. Let $W(x, x_0)$ be the polar form of W. The function

$$u=W(x, x_0)W^{-\frac{1}{2}}(x)W^{-\frac{1}{2}}(x_0)$$

is then the same for all choices of y_1, y_2 and is called by the author the monodromic function for the differential equation. If in the original equation y is replaced by $(u''-pu)/p'$, there results an equation, called here a resolvent, which has a likeness to a Galois resolvent. The solution of the resolvent satisfying the initial conditions $u(x_0)=1$, $u'(x_0)=0$, $u''(x_0)=p(x_0)$, $u'''(x_0)=\frac{3}{2}p'(x_0)$ is the monodromic function.

J. M. Thomas (Durham, N. C.).

Matsumoto, Toshizô. A note on Fowler's differential equation. II. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **24**, 93–97 (1944).

[For part I cf. the same vol., 79–81 (1944); these Rev. **7**, 297.] It was shown by Sansone by means of an identity involving $\int_0^t t^2y'^2dt$ that a necessary and sufficient condition for the existence of a unique solution of $(t^2y')'+t^ay^n=0$, vanishing at a finite point, and satisfying the initial condition $y(0)=c_1>0$, is $2a-n+1>0$. The author establishes a corresponding identity for $\int_0^t t^by'^2dt$, where y is a solution of $(t^by')'+ct^ay^n=0$, and thereby obtains a generalization of the Sansone result. *R. Bellman.*

★**Infeld, Leopold. The factorization method and its application to differential equations in theoretical physics.** Proc. Symposia Appl. Math. **2**, 58–65 (1950). $3.00.

A report on the author's factorization method for the solution of second-order Sturm-Liouville eigenproblems, published previously [Physical Rev. (2) **59**, 737–747 (1941); these Rev. **2**, 364; for a generalization for non-degenerate cases see Trans. Roy. Soc. Canada. Sect. III. (3) **36**, 7–18 (1942); these Rev. **4**, 144]. The present account does not add anything basically new, but stresses the historical context and mechanical aspects of the method, in particular, its rapidity as compared to conventional methods. There are eleven different types of differential equations in which the method yields closed solutions ("pure cases"). The factorization method extends to perturbation procedures starting with pure cases. The theoretical aspects of the method have not yet been explored; it is suggested that they lie in the realm of group theory. *H. G. Baerwald.*

Duff, G. F. D. Factorization ladders and eigenfunctions. Canadian J. Math. **1**, 379–396 (1949).

The factorization method replaces the second order equation

$$u''+r(x, m)+\lambda u=0, \qquad a\leqq x\leqq b,\ m=0, 1, 2, \cdots,$$

and its associated boundary conditions

$$u(a)=0, \quad u(b)=0, \quad \int_a^b u^2dx=1,$$

by a system of two first order equations

$$(*)\qquad \begin{aligned} &\left\{k(x, m+1)-\frac{d}{dx}\right\}u_l^m(x)=A_l^{m+1}u_l^{m+1}(x),\\ &\left\{k(x, m+1)+\frac{d}{dx}\right\}u_l^{m+1}(x)=A_l^{m+1}u_l^m(x),\end{aligned}$$

where u_l^m, u_l^{m+1} are contiguous eigenfunctions belonging to the eigenvalue $\lambda_l=L(l+1)$, $l=0, 1, 2, \cdots$;

$$A_l^{m+1}=[L(l+1)-L(m+1)]^{\frac{1}{2}}$$

is a constant, which vanishes when $m=l$. The author supposes that corresponding to $m=l$ or $l+1$ there exists a normalized eigenfunction $C\exp\{\pm\int_{x_0}k(r, l+1)dr\}$. Then for neighbouring integral values of m there exist normalized eigenfunctions given by (*), provided that the corresponding constants A_l^m are real and different from zero. If $L(m)$ is an increasing function of m ($m\geqq 0$) then the permitted values of l and m are $l=0, 1, 2, 3, \cdots$; $m=0, 1, \cdots, l$. The finite ladder of eigenfunctions $u_l^0, u_l^1, \cdots, u_l^l$ is called an Infeld ladder [cf. Physical Rev. (2) **59**, 737–747 (1941); these Rev. **2**, 364]. If $L(m)$ is a decreasing function of m for $m\geqq 0$ then the permitted values for l and m are $l=0, 1, 2, \cdots$; $m=l, l+1, l+2, \cdots$. The author calls this infinite ladder of eigenfunctions a Schrödinger ladder, as Schrödinger used infinite ladders in the solution of the Kepler problem. In the "An Essay Toward a Unified Theory of Special Functions . . ." of C. A. Truesdell [Princeton University Press, 1948; these Rev. **9**, 431] relations of the form

$$\frac{\partial}{\partial y}f(y, \alpha)=A(y, \alpha)f(y, \alpha)+B(y, \alpha)f(y, \alpha+1)$$

are discussed and a "reducibility condition" on A and B is found, which allows this equation to be transformed into the "F-equation"

$$\frac{\partial}{\partial z}F(z, \alpha)=F(z, \alpha+1).$$

The recurrence formulae (*) are reducible to the F-equation if and only if $k(x, m)=mk(x)+k_0(x)$, i.e., the reducibility condition for the factorization method is that $k(x, m)$ is linear in m. The author proves: any solution $F(z, m)$ of the equation

$$\frac{d}{dz}F(z, m)=F(z, m+1)$$

satisfies

$$\frac{d^n}{dz^n}F(z, m)=F(z, m+n) \qquad (n=0, 1, 2, \cdots).$$

If $k(x, m)$ is linear in m two reducible recursion formulae exist in the case of an Infeld ladder. The author mechanizes the construction of the eigenfunctions in two different ways. One of these is illusory, because there is no independent

starting point for it. The types

$$k(x, m) = \alpha(m+\gamma)\cot \alpha x + C/\sin \alpha x, \quad 0 \leqq x \leqq \tfrac{1}{2}\pi;$$
$$k(x, m) = (m+\gamma)/x + bx/2, \quad 0 \leqq x < \infty, \quad b < 0;$$
$$k(x, m) = -x, \quad -\infty \leqq x \leqq +\infty$$

are treated thoroughly. Other examples are solutions $k(x, m)$ corresponding to $L(m)$ a decreasing function of m (eigenfunctions on a Schrödinger ladder):

$$k(x, m) = Ce^{\beta x} - \beta(m+\gamma), \qquad -\infty \leqq x < \infty,$$
$$k(x, m) = \beta(m+\gamma)\coth \beta x + \frac{C}{\sinh \beta x}, \qquad -\infty < x \leqq 0,$$
$$k(x, m) = \beta(m+\gamma)\tanh \beta x + \frac{C}{\cosh \beta x}, \qquad -\infty < x < \infty.$$

Finally in the last section the author uses a device which transforms each of the irreducible types into one of the known reducible problems belonging to the opposite class. A correspondence between the Infeld and Schrödinger ladders is thus established. In all cases treated the author is able to find the normalized eigenfunctions explicitly.

M. Pinl (Dacca).

Povzner, A. On differential equations of Sturm-Liouville type on a half-axis. Amer. Math. Soc. Translation no. **5**, 79 pp. (1950).

Translated from Mat. Sbornik N.S. **23**(65), 3–52 (1948); these Rev. **10**, 299.

Fel'dbaum, A. A. Integral criteria for the quality of a regulation. Avtomatika i Telemehanika **9**, 3–19 (1948). (Russian)

Consider the system $dx/dt = Ax$, where A is a constant matrix whose characteristic roots have negative real parts. Let $V = x'Bx$ be a Liapounoff form, i.e., $dV/dt < 0$. In connection with automatic regulators, the integral $\int_0^\infty V dt$ is of importance. The author discusses various examples derived from electrical systems.

R. Bellman.

Bulgakov, B. V., and Litvin-Sedoĭ, M. Z. On a problem of automatic regulation with nonlinear characteristic. Avtomatika i Telemehanika **10**, 329–341 (1949). (Russian)

Let φ, θ be coordinates for the basic system to be regulated and ζ a coordinate for the regulator (servomotor). Let the basic differential equations be

$$T^2\ddot{\varphi} + U\dot{\varphi} + k\theta + \zeta = 0,$$
$$S\dot{\theta} + \theta = S\dot{\varphi},$$
$$V^2\ddot{\zeta} + W\zeta = H(\psi),$$
$$Y^2\ddot{\varphi} + X\dot{\varphi} + l\varphi - \zeta/m = \psi.$$

Here $H(\psi)$, the characteristic of the servomotor, is a nonlinear function very close to a broken line made up of a vertical segment upwards from the origin followed by an infinite horizontal ray to the right. The other latin letters are constants, all positive except X and k, and their physical significance is more or less indicated by the equations. Since $H(\psi)$ is small the author replaces it by another function written $\mu H(\psi)$, μ small. He then applies the perturbation method and discusses the self-oscillations of the system and their amplitudes and stability for μ small. The effects produced by varying the parameters of the servomotor are also considered [see B. V. Bulgakov, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 263–280 (1942); **10**, 313–332 (1946); these Rev. **4**, 142; **8**, 207].

S. Lefschetz (Princeton, N. J.).

Sears, D. B. On the solutions of a linear second order differential equation which are of integrable square. J. London Math. Soc. **24**, 207–215 (1949).

The equation (*) $y'' + (\lambda - q(x))y = 0$ is considered over the finite interval $0 < x \leqq a$ with $q(x)$ continuous over the same interval. If, for some constant A and small x, $q(x) > A + 3/(4x^2)$, then (*) has, for any λ, a solution $\phi(x, \lambda)$ for which $\liminf x^{\frac{1}{2}}|\phi(x, \lambda)| > 0$ as $x \to +0$. The constant 3/4 is best possible. A number of other results are given.

N. Levinson (Cambridge, Mass.).

Kučer, D. L. On some criteria for the boundedness of the solution of a system of differential equations. Doklady Akad. Nauk SSSR (N.S.) **69**, 603–606 (1949). (Russian)

The author considers the problem of determining the necessary and sufficient conditions that must be imposed upon the solutions of the vector-matrix equation $dy/dt = A(t)y$ in order that all solutions of $dz/dt = A(t)z + f(t)$ be bounded for vector functions $f(t)$ belonging to different function spaces such as L, L_p, and M. The first results of this type were obtained by Perron [Math. Z. **32**, 465–473 (1930)] for the space M, using matrix transformation methods. A direct method was employed by Malkin [Sbornik Nauchnyuh Trudov Kazanskogo Aviatzionnogo Instituta, no. 7 (1937)], who showed that $\|y(t)\| \leqq c_1 \exp(-c_2(t-t_0))\|y(t_0)\|$, $t \geqq t_0$, for all $t_0 \geqq 0$, c_1, c_2, fixed constants, was necessary and sufficient for $f \varepsilon M$. Banach space methods were introduced by Bellman [Ann. of Math. (2) **49**, 515–522 (1948); these Rev. **10**, 121] to extend Perron's result to L_p, and to difference and allied functional equations. Kučer reverts to Malkin's method to show that the criteria of Bellman may be put into much simpler form, akin to that of Malkin's above, from which more information can be obtained concerning the solutions of $dy/dt = A(t)y$.

R. Bellman (Stanford University, Calif.).

Ważewski, T. Sur les intégrales asymptotiques des équations différentielles ordinaires. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 38–42 (1948). (French. Polish summary)

The author presents a topological method which he states can be used to discuss the asymptotic behavior or the oscillatory behavior of a solution in the neighborhood of a singular point. An example is given of the application of the method.

R. Bellman (Stanford University, Calif.).

Bielecki, Adam. Sur certaines conditions nécessaires et suffisantes pour l'unicité des solutions des systèmes d'équations différentielles ordinaires et des équations au paratingent. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **2** (1947), 49–106 (1948). (French. Polish summary)

The system considered is $x' = f(t, x, y)$, $y' = g(t, x, y)$, where f, g are continuous in $a \leqq t \leqq b$, $-\infty < x < \infty$, $-\infty < y < \infty$, with a particular solution I_0 defined in $a \leqq t \leqq b$ and reducing to (ξ, η) for $t = a$. This paper establishes as a necessary and sufficient condition for the uniqueness of the solution with the given initial conditions the existence of a closed set C with the following properties: (i) C contains I_0 and is contained in a neighborhood of I_0; (ii) there are two cones with vertex $P_0 = (a, \xi, \eta)$, one of which contains the tangent to I_0 at P_0 in its interior and is locally contained in C, and the other of which locally contains C; (iii) every solution with initial point on the boundary of C penetrates the interior of C for increasing values of t; (iv) the common part of the boundary of C and the region $a < t < b$ is a regular surface Σ; (v) the intersections of Σ and $t = c$ for $a \leqq c \leqq b$ are simple circuits.

J. M. Thomas (Durham, N. C.).

Cinquini, Silvio. Sopra il metodo della scuola dell'Arzelà per il problema di Nicoletti per le equazioni differenziali ordinarie. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 173–188 (1949).

Scorza Dragoni, Giuseppe. Ancora a proposito di alcuni teoremi sulle equazioni differenziali ordinarie. Rend. Sem. Mat. Univ. Padova 18, 115–139 (1949).

Continuation of a discussion between the authors; cf. Scorza Dragoni, same Rend. 15, 60–131 (1946); 16, 1–2 (1947); these Rev. 8, 207; 9, 353.

Chiellini, Armando. Sui sistemi differenziali lineari ordinari e sui loro aggiunti di Lagrange. II. Pont. Acad. Sci. Acta 13, 113–128 (1949).

[For the notation cf. the review of the first part of this paper, same vol., 9–26 (1949); these Rev. 11, 179]. The system (1) is termed a reduced or a semicanonical one, respectively, if $\Gamma^i_{i1}=0$ (not summing for i) or $\Gamma^i_{j1}=0$ ($i, j=1, \cdots, m$), respectively. A transformation $y^i=\lambda^i Y^i$ or $y^i=\sum_{k=1}^m \lambda_k{}^i Y^k$, respectively, brings the given system into a reduced or semicanonical form, respectively, provided the λ's are solutions of a system of linear differential equations. The invariants θ [cf. the author, same Acta 12, 199–213 (1948); these Rev. 11, 251] of the system (1) and (2) are equal or differ only by sign if their weight is even or not. These results enable the author to find necessary and sufficient conditions by means of invariants θ and ϑ for a system (1) to be self-adjoint. The last section is devoted to systems with $\vartheta_2{}^{(i)}=0$. *V. Hlavatý* (Bloomington, Ind.).

Tsuji, Masatsugu. On a system of total differential equations. Jap. J. Math. 19, 383–393 (1948).

Let S be a system specifying the first partial derivatives of m unknowns z_i with respect to n independent variables x_j as given functions $a_{ij}(z, x)$ which in $|x_j|<a$, $|z_i|<b$ (i) are continuous and uniformly bounded, (ii) are totally differentiable in Stolz' sense, (iii) have uniformly bounded first partial derivatives with respect to the z's, and (iv) satisfy the integrability conditions. Then S has in $|x_j|<\delta\leqq a$ a unique solution vanishing for $x_j=0$. The author points out that W. Nikliborc [Studia Math. 1, 41–49 (1929)] proved the conclusion assuming instead of (ii) and (iii) the uniform continuity of all the first derivatives of the a's. *J. M. Thomas* (Durham, N. C.).

See the note to p. 241 Tsuji

Pini, Bruno. Sui sistemi di infinite equazioni lineari del primo ordine ai differenziali totali. Giorn. Mat. Battaglini (4) 2(78), 151–167 (1949).

The author discusses the vector-matrix total differential equation of infinite order given by

$$dy/dx=\sum A_i(x_1, x_2, \cdots, x_n)ydx_i,$$

considering first the case $n=1$, and then the general case. Under different assumptions concerning the matrix A_i, this case has been treated by various authors [cf. Hart, Amer. J. Math. 39, 407–424 (1917); Cimmino, Mem. Accad. Naz. Lincei. Cl. Sci. Fis. Mat. Nat. (6) 5, 273–315 (1933); Arley and Borchsenius, Acta Math. 76, 261–322 (1945); these Rev. 7, 161; Bellman, Duke Math. J. 14, 695–706 (1947); these Rev. 9, 145]. The existence of a solution of finite norm in Hilbert space for all $t\geqq 0$ is established under the assumption that A is a bounded matrix. The method is that of conversion of the differential equation into an integral equation. *R. Bellman* (Stanford University, Calif.).

Cronin, Jane. The existence of multiple solutions of elliptic differential equations. Trans. Amer. Math. Soc. 68, 105–131 (1950).

The author is concerned with the existence and multiplicity of solutions $z(x, y)$ neighboring a given initial solution $z_0(x, y)$ of a nonlinear differential equation $F(x, y, z, p, q, r, s, t)=z_0(x, y)$, where z_0 is "elliptic relative to F" and (x, y) ranges over a closed circular disc. The earlier classical work [Encyklopädie der Math. Wiss., v. 3.2, part II] assumed that the associated Jacobi equation had only the null solution, or one linearly independent solution. This type of assumption is here replaced by a less restrictive topological one involving a "multiplicity" of solutions, related to but not identical with the topological degree of Leray and Schauder [Ann. Sci. École Norm. Sup. (3) 51, 45–78 (1934)]. Earlier topological techniques are supplemented by results on equations in Banach space due to the author. When the elliptic equation is quasi-linear the author's multiplicity has the properties of a topological degree. Existence theorems are obtained in a variety of cases with special attention to the number of solutions. *M. Morse* (Princeton, N. J.).

Picone, Mauro. Intorno alla teoria di una classica equazione a derivate parziali della fisica-matematica. Ann. Soc. Polon. Math. 21 (1948), 161–169 (1949).

Soit dans l'espace à $n\geqq 2$ dimensions, le domaine borné Ω de diamètre δ, et un nombre fini de points-frontière P_i auxquels sont associés des coefficients $\alpha_i>0$ de somme α avec $\alpha<n-2$ si $n>2$ et $\alpha<1$ si $n=2$. On considère une solution u de l'équation $\Delta u+c(M)u=0$ où c est inférieur à $\alpha(n-2-\alpha)/\delta^2$ ($n>2$) ou à $\alpha(1-\alpha)/\delta^2$ ($n=2$). Alors si $u\to 0$ en tout point-frontière différent des P_i et si en chaque P_i, pour $P\to P_i$, $u(P)\overline{P_iP}^{\alpha_i}\to 0$ ($n>2$) ou bien $u(P)(\log 1/P_iP)^{-\alpha_i}\to 0$ ($n=2$), u est nulle. Pour la démonstration l'auteur introduit la fonction, par exemple si $n>2$, $\omega=\prod_i \overline{P_iP}^{\alpha_i}$ et montre par des calculs élémentaires et simples que $\Delta\omega+c\omega<0$, d'où résulte que u/ω ne peut admettre d'extremum. L'auteur fait ensuite quelques applications. Quant au théorème principal, tout en soulignant l'intérêt d'un c non nécessairement $\leqq 0$, il faut signaler que l'on sait depuis longtemps établir le résultat pour c continue $\leqq 0$ mais par contre avec l'hypothèse meilleure sur les α_i: $\alpha_i=1$ ($n=2$), $\alpha_i=n-2$ ($n>2$). Il n'y a qu'à adapter un raisonnement donné par Zaremba dans le cas harmonique. [Brelot l'a explicité pour $c<0$ et $n=2$, Rend. Circ. Mat. Palermo 55, 21–49 (1931).] *M. Brelot* (Grenoble).

Leja, F. Remarques sur le travail précédent de M. Mauro Picone. Ann. Soc. Polon. Math. 21 (1948), 170–172 (1949).

L'auteur améliore le théorème principal de Picone [voir l'analyse précédente] en retouchant le raisonnement. Ainsi on peut remplacer les α_i par un même nombre $\alpha>0$ ($\alpha<n-2$ si $n>2$, $\alpha<1$ si $n=2$), ce nombre α intervenant de la même manière que précédemment dans la limitation de c. *M. Brelot* (Grenoble).

Titchmarsh, E. C. Eigenfunction expansions for a finite two-dimensional region. Quart. J. Math., Oxford Ser. 20, 238–253 (1949).

Let D be a simply-connected region in the (x, y)-plane bounded by a finite number of curves which are either of the form $y=F(x)$, where $F(x)$ satisfies a Lipschitz condition of order one, or of a similar form with x and y interchanged.

The author exhibits a complete set of eigenfunctions for the equation $u_{xx}+u_{yy}+\{\lambda-q(x,y)\}u=0$ under the condition $u=0$ on the boundary of D as limits of the eigenfunctions of the sequence of equations $u_{xx}+u_{yy}+\{\lambda-q_\nu(x,y)\}u=0$ associated with a square B containing D, where the functions $q_\nu(x,y)$ are so defined that $q_\nu(x,y)=q(x,y)$ for $(x,y)\varepsilon D$ and $q(x,y)\to+\infty$ for $(x,y)\varepsilon B-D$. A Green's function is also obtained in this way. *I. I. Hirschman, Jr.*

✱Campbell, James Dow. The Parametric Theory of Singular Parabolic Partial Differential Equations. Abstract of a thesis, University of Illinois, 1946. ii+11 pp.

The parabolic equations considered are of the form

$$(1)\qquad L(s)=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 s}{\partial x_i\partial x_j}+\sum_{i=1}^{n}\lambda^H a_i\frac{\partial s}{\partial x_i}+\lambda^{2H}as-\frac{\partial s}{\partial y}=0.$$

Here the a_{ij} are coefficients of a positive definite form. The a_{ij}, a_i, a are functions of the x_i and y. In addition the a_i and a are functions of the parameter λ, and are assumed to have asymptotic expansions according to powers of $1/\lambda$. Following a method of Trjitzinsky [Rec. Math. [Mat. Sbornik] N.S. **15**(57), 179–242 (1944); these Rev. **6**, 231] "formal" solutions of (1) can be obtained, which are of the form $s=e^{Q(x,y,\lambda)}\sigma(x,y,\lambda)$, where Q is a polynomial of degree H in λ, and σ a formal power series in $1/\lambda$. The author shows that under certain conditions there are "actual" solutions of (1), that behave asymptotically up to a certain order like the formal solutions. These actual solutions are obtained by solving an appropriate integral equation in the manner used by Dressel [Duke Math. J. **7**, 186–203 (1940); these Rev. **2**, 204] for construction of fundamental solutions of parabolic equations. *F. John* (New York, N. Y.).

Traupel, W. Unsteady heat conduction in plates, cylinders and spheres. Sulzer Tech. Rev. **1949**, no. 3, 12–23 (1949).

Temperatures $T(r,t)$ in a long cylinder subjected to linear heat transfer at its surface $r=a$ from surroundings whose temperature is a prescribed function of time t are considered and applied to a problem in the heating of a drum-type turbine rotor. The results indicate that maximum temperature differences in the rotor are only moderately affected by assuming that the surrounding gas immediately assumes its maximum temperature. The well-known Duhamel formula is derived again in the theoretical part of the paper. There is also a discussion of corresponding problems for slabs and spheres, and a number of graphs of temperature functions. *R. V. Churchill* (Ann Arbor, Mich.).

Benfield, A. E. The temperature in an accreting medium with heat generation. Quart. Appl. Math. **7**, 436–439 (1950).

A semi-infinite medium $x>0$ moves with a constant velocity v in the positive direction of the x-axis and the material at the face is replenished at a constant rate so that this face maintains a fixed position $x=0$. The initial temperature of the medium and the temperature of the new material have the same constant value; also all the material generates heat at a constant rate. The temperature $T(x,t)$ then satisfies a differential equation of the type $T_t=kT_{xx}-vT_x+a$, where k and a are constants. The boundary value problem in $T(x,t)$ is solved here, using Laplace transforms, to obtain a simple formula for $T(x,t)$ in terms of error functions. *R. V. Churchill* (Ann Arbor, Mich.).

Cattaneo, Carlo. Sulla conduzione del calore. Atti Sem. Mat. Fis. Univ. Modena **3**, 83–101 (1949).

The author points out that certain solutions given to boundary value problems associated with the heat equation imply that heat is conducted instantaneously over great distances. He feels this cannot be the physical situation and argues that the temperature does not satisfy a parabolic, but rather a hyperbolic, partial differential equation. *F. G. Dressel* (Durham, N. C.).

Sestini, Giorgio. Sopra la conduzione del calore in una piastra sottile limitata da due circonferenze concentriche. Atti Sem. Mat. Fis. Univ. Modena **3**, 125–137 (1949).

Let (r,θ) be polar coordinates, $f(r)$, $\phi_1(t)$, $\phi_2(t)$ given continuous functions, and a and b constants. The following boundary value problem for a circular disk is treated using the Laplace transformation:

$$(1)\qquad T_{rr}+r^{-1}T_r-T_t=aT-b,\qquad 0<r_1<r<r_2;\ t>0,$$

$$\lim_{t\to 0}T(r,t)=f(r),\quad T(r_1,t)=\phi_1(t),\quad T(r_2,t)=\phi_2(t),\quad t>0.$$

The Laplace transformation is also used to construct a Green's function for the above boundary value problem. Without further assumptions on T the author's uniqueness "proof" must be considered questionable. *F. G. Dressel* (Durham, N. C.).

Bononcini, Vittorio E. Un problema della propagazione del calore. Atti Sem. Mat. Fis. Univ. Modena **3**, 142–161 (1949).

Let S be the rectangular region $0<x<L$, $0<y<l$, and let C be the boundary of S. The author uses the method of separation of variables to solve the following two boundary value problems. In problem one, u is a solution in S of the equation

$$(1)\qquad Ku_t=u_{xx}+u_{yy}+\lambda-bu$$

(λ, b constants), with $K=0$, and $u=0$ on C. In problem two u is a solution of (1) in S for $t>0$ and $K=1$, meeting the boundary conditions $u(x,y,t)=0$, $t=0$, (x,y) in $S+C$; $u(x,y,t)=0$, $t>0$, (x,y) in C. *F. G. Dressel.*

Davies, C. N. The sedimentation and diffusion of small particles. Proc. Roy. Soc. London. Ser. A. **200**, 100–113 (1949).

Let $c(x,t)$ denote the concentration of particles suspended in a horizontal layer $(0\leqq x\leqq 1)$ of fluid. When the particles reach the lower surface $x=1$ by diffusion and sedimentation they adhere to that surface. There is no transfer of particles to the surface $x=0$. Initially the concentration is uniform. The boundary value problem in $c(x,t)$ then has the form $c_t=\alpha c_{xx}-c_x$, $c(x,0)=1$, $c(1,t)=0$, $\alpha c_x(0,t)=c(0,t)$, where the constant α depends upon the coefficients of diffusion and sedimentation. The problem is first solved by separating variables and using an elementary Sturm-Liouville expansion. Also, a formal procedure is used to get an approximate expression for $c(x,t)$ in terms of error functions, a form of solution that is generally obtained easily by using Laplace transforms. The two forms of solutions are written also for the more elementary problem obtained by replacing the last boundary condition by the condition $c(0,t)=0$. A table of roots of the characteristic equation $\tan p=-2\alpha p$ and graphs of the concentration function are presented. *R. V. Churchill* (Ann Arbor, Mich.).

Fumi, F., e Carrassi, M. Integrazione della equazione delle corde vibranti nel vuoto per mezzo degli operatori funzionali in una variabile. Atti Accad. Ligure **5**, 42–67 (1949).

A detailed discussion by means of operational methods of the partial differential equation $u_{xx}-a^{-2}u_{tt}=0$ of the vibrating string. The only difference from the presentation of this subject in text books is that the formal application of the Heaviside calculus is justified by the theory developed by Sbrana and the first named author [same vol., 7–33 (1949); Sbrana, ibid., 173–186, 187–200, 201–217 (1949); Boll. Un. Mat. Ital. (3) **4**, 34–40 (1949); these Rev. **10**, 701, 702] rather than by Laplace transform methods.

A. Erdélyi (Pasadena, Calif.).

Romero Juarez, Antonio. Propagation of waves of finite amplitude. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1947**, 97–103 (1949). (Spanish)

The author considers the one-dimensional equation of motion (Eulerian form) $y_{tt}=-\rho^{-1}p_y$ under the assumption that $p/\rho^\gamma=p_0/\rho_0^\gamma$. Employing the continuity equation, and using Lagrangian variables x and t, the equation of motion becomes $\rho_0p_0^{-1}\gamma^{-1}y_{tt}=y_x^{-\gamma-1}y_{xx}$. Two particular solutions (waves of finite amplitude) of this equation are discussed.

J. B. Diaz (Providence, R. I.).

Faedo, Sandro. Un nuovo metodo per l'analisi esistenziale e quantitativa dei problemi di propagazione. Ann. Scuola Norm. Super. Pisa (3) **1** (1947), 1–41 (1949).

Ritz's method of solving a linear self-adjoint partial differential equation $L[u]=f$ of the second order and elliptic type in a domain C, under the boundary condition $u=0$ on the frontier of C, can be expressed as follows: let $\Phi_i(P)$ be a complete orthonormal system of functions defined in C and zero on the frontier of C, and let $u_n(P)=\sum_{i=1}^n c_{i,n}\Phi_i(P)$; then, if the constants $c_{i,n}$ are chosen to satisfy the system of linear equations

$$\int_C\{L[u_n]-f\}\Phi_i(P)dC=0, \qquad i=1,\cdots,n,$$

the function $u_n(P)$ converges to the desired solution, as $n\to\infty$, under quite general conditions. In the case of a problem of wave-propagation, u depends not only on the position of P in C but also on the time t. In this case L is an operator of hyperbolic type, and, in addition to the boundary condition $u=0$, there are initial conditions $u(P,t)=g(P)$, $u_t(P,t)=g_1(P)$ when $t=0$. The Ritz procedure can be modified as follows. With the same orthonormal set $\Phi_i(P)$, we write $u_n(P,t)=\sum_{i=1}^n c_{i,n}(t)\Phi_i(P)$, and we choose the functions $c_{i,n}(t)$ to satisfy the system of ordinary differential equations

$$\int_C\{L[u_n]-f\}\Phi_i(P)dC=0, \qquad i=1,\cdots,n,$$

under the initial conditions

$$c_{i,n}(0)=\int_C g(P)\Phi_i(P)dC, \quad c'_{i,n}(0)=\int_C g_1(P)\Phi_i(P)dC.$$

Faedo calls this the method of moments. The question is whether $u_n(P,t)$ converges to the solution. In the present paper the method is applied to the equation

$$L[u]\equiv-u_{xx}+a_1u_{xt}+a_2u_{tt}+a_3u_x+a_4u_t+a_5u=f$$

where the coefficients $a_1,\cdots,a_5$ are functions of x and t, and $a_2>0$. The domain C is now a linear interval $0\leqq x\leqq c$. Theorems of uniqueness, existence and convergence are proved under quite general conditions.

E. T. Copson.

Faedo, Sandro. Un nuovo metodo per l'integrazione dei problemi di propagazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 435–438 (1949).

A statement of results proved in the paper reviewed above.

E. T. Copson (Dundee).

Casulleras Regás, Juan. Application of the theory of analytic functionals to the solution of a type of partial differential equations of the third order. Collectanea Math. **1**, 3–60 (1948). (Spanish)

L. Fantappiè [Ann. Mat. Pura Appl. (4) **22**, 181–289 (1943); these Rev. **8**, 589], as an application of his theory of analytic functionals, has given a general method for the solution of Cauchy's problem for linear partial differential equations of any order, with constant coefficients. The solution is expressed as a partial derivative of a "prodotto funzionali proiettivo" [see p. 286 of the paper quoted above]. The present author applies Fantappiè's method to treat Cauchy's problem for the linear equation of third order with constant coefficients,

$$\sum_{k+r\leqq 3} a_{kr}\partial^{k+r}z(t,x)/\partial t^k\partial x^r=f(t,x).$$

Without loss, the equivalent homogeneous equation

$$(1)\qquad \sum_{k+r+s=3} a_{krs}\partial^3u(t,x,y)/\partial t^k\partial x^r\partial y^s=F(t,x,y),$$

obtained by letting $u(t,x,y)=e^y\cdot z(t,x)$; $F(t,x,y)=e^y\cdot f(t,x)$, is considered throughout. Regarding (t,x,y) as nonhomogeneous coordinates in four-dimensional projective space, each characteristic plane of (1) intersects the improper plane in a straight line, and these straight lines envelop a certain algebraic curve in the improper plane. Equations of type (1) fall into equivalent classes, according to whether their corresponding algebraic curves in the improper plane are projectively equivalent or not in that plane, and the solution of Cauchy's problem is shown to reduce to its solution for a single representative equation out of each equivalence class. The author considers, in particular, the equivalence class represented by a cardioid in the improper plane, and treats the corresponding Cauchy problem, which involves the equation

$$4u_{ttt}+12u_{ttx}-15u_{txx}-27u_{tyy}+4u_{xxx}=F.$$

The Cauchy problem for this last equation, the Cauchy data being given on a non-characteristic surface Γ, is solved by Fantappiè's method, by first expressing the solution in terms of a certain projective functional product. The actual evaluation of this functional product occupies the major portion of the paper.

J. B. Diaz (Providence, R. I.).

Hodyreva, V. M. On a minimal property of the circle. Doklady Akad. Nauk SSSR (N.S.) **69**, 615–618 (1949). (Russian)

Consider the proper value problem determined by the equation $(-1)^k\Delta^ku-\lambda u=0$, where

$$\Delta^k=\sum_{m=1}^k\binom{k}{m}\frac{\partial^{2k}}{\partial x^{2(k-m)}\partial y^{2m}},$$

on a plane region G with a piecewise smooth boundary curve in an interior neighborhood of which the solution is supposed to vanish together with its derivatives of order not exceeding $k-1$. The author proves that for all such regions the

first proper value is a minimum for the circle. This proper value is the minimum of

$$\iint_G \sum_{m=0}^{m=k} \binom{k}{m} \left(\frac{\partial^k u}{\partial x^{k-m} \partial y^m}\right)^2 dxdy$$

with the side condition $\iint u^2 dxdy = 1$. The method, similar to that of Courant [Math. Z. 1, 321–328 (1918)], is one of symmetrization; by use of analytic continuation by reflection a minimal sequence of polygons is shown to consist of regular polygons which converge to a circle. Use is made of the facts that the first proper value is simple and the first proper function does not change sign in the region, of the continuous variation of the proper values with the region, and that if A is a proper subset of the interior B then the first proper value belonging to the region A is greater than that of region B. *M. J. Gottlieb* (Chicago, Ill.).

Difference Equations, Special Functional Equations

Krull, Wolfgang. Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$. II. Math. Nachr. 2, 251–262 (1949).

In a previous paper with the same title [Math. Nachr. 1, 365–376 (1948); these Rev. 11, 112] the author has shown that a solution g satisfying a certain linear boundary condition at ∞ of the difference equation $g(x+1)-g(x)=\varphi(x)$ is uniquely determined up to an additive constant provided φ is in a certain linear class of functions. Two explicit representations of the solution were given. In the present paper the same program is carried through for an essentially wider linear class of functions: namely the class of functions φ which for some $r=0, 1, 2, \cdots$ possess an rth derivative of bounded variation near ∞. The proof proceeds by induction on r. [Formula (3) of the paper is incorrect but can be corrected.] *W. Gustin* (Bloomington, Ind.).

Strodt, Walter. On a class of nonlinear difference equations in the complex domain. Trans. Amer. Math. Soc. 68, 132–164 (1950).

The equation considered has the form

$$(1) \qquad \Lambda(x, y(x+\omega_1), \cdots, y(x+\omega_n))=0,$$

where $\Lambda(x, y_1, \cdots, y_n)$ is a polynomial of degree $d \geqq 1$ in $y_1, \cdots, y_n$ with coefficients which are admissible functions of x. The constant $\omega_1=0$ and the remaining constants $\omega_2, \cdots, \omega_n$ are not zero and satisfy the condition $|\arg \omega_k| < \beta \leqq \pi/2$, $k=2, \cdots, n$. The function $\varphi(x)$ is called admissible if it fulfills the conditions (a) for every positive number $\gamma < \beta \leqq \pi/2$ there is a positive number ξ such that $\varphi(x)$ is analytic and bounded in the sector $\{x; x \neq \xi, |\arg(x-\xi)| < \gamma\}$, where $-\pi < \arg x \leqq \pi$; (b) $\varphi(x)$ approaches a limit as x becomes infinite along the positive real axis. The principal theorem states that the totality $\mathcal{T}$ of all admissible solutions of (1) is the union $\mathcal{T}_1+\cdots+\mathcal{T}_c$, where $\mathcal{T}_k$ is the totality of admissible solutions of (1) such that $y(+\infty)=\sigma_k$, $k=1, 2, \cdots, c$, and the σ_k are the zeros of the polynomial $\Lambda(+\infty, \sigma, \cdots, \sigma)$, which is assumed to be not identically zero and to have only simple zeros. Each $\mathcal{T}_k$ is a nonempty finite-parameter family of functions with a number of parameters equal to the finite number of zeros of the exponential polynomial $f_k(z) \equiv \sum_{j=1}^n a_j(+\infty, \sigma_k, \cdots, \sigma_k)e^{\omega_j z}$; $a_j(x, y_1, \cdots, y_n) = \partial\Lambda(x, y_1, \cdots, y_n)/\partial y_j$ which lie in the complement of the sector $\{z; z \neq 0, |\arg z| < \beta+\pi/2\}$. Further, $\mathcal{T}$ contains not only all admissible solutions of (1) but also all solutions of (1) required to satisfy only condition (a) above.

The method of proof employs a successive approximation process depending upon the solution of linear nonhomogeneous difference equations with constant coefficients by methods related to those developed in an earlier paper of the author [same Trans. 64, 439–466 (1948); these Rev. 10, 303]. The present paper extends earlier results of the author on principal solutions of difference equations [Amer. J. Math. 69, 717–757 (1947); these Rev. 9, 289]. In conclusion the general results are illuminated by a number of detailed examples. *P. E. Guenther* (Cleveland, Ohio).

Wright, E. M. Perturbed functional equations. Quart. J. Math., Oxford Ser. 20, 155–165 (1949).

Let $\Lambda\{y(x)\}$ denote $\sum_{j=0}^n \int_0^b y^{(j)}(x-\xi)dk_j(\xi)$, where each k_j is the sum of a step function with a finite number of discontinuities and an absolutely continuous function. The author considers the stability of the perturbed equation (*) $\Lambda(y)=\Omega(x, y)$, where Ω depends on x and on $y^{(j)}(x-\xi)$, $0 \leqq \xi \leqq b$ and $0 \leqq j < n$. The function Ω is smaller than linear in $|y^{(j)}|$ for small $|y^{(j)}|$ and satisfies a Lipschitz condition. It is shown that if the characteristic function associated with Λ has no zeros in the $s=\sigma+it$ plane to the right of $s=\sigma_1$ and if $\sigma_1<0$ then any solution of (*) which is initially small tends to zero as $x \to \infty$. Results of Pitt [Proc. Cambridge Philos. Soc. 40, 199–211 (1944); 43, 153–163 (1947); these Rev. 6, 273; 9, 40] for the linear equation $\Lambda(y)=v(x)$ play a fundamental role. Other results are obtained. The known results for differential equations are a very special case of this comprehensive result. *N. Levinson.*

***Minorsky, N. On certain applications of difference-differential equations.** Departments of Engineering and Mathematics, Stanford University, Stanford, Calif., 1948. 38 pp.

The author treats the nonlinear differential-difference equation,

$$\ddot{x}(t)+a\dot{x}(t)+b\dot{x}(t-h)+cx(t)+dx(t-h)=f(x(t), x(t-h)).$$

Equations of this special type occur in many important applications where time lags and retarded actions are involved. He first emphasizes the important distinction between differential-difference equations and ordinary differential equations. The characteristic equation of an ordinary differential equation is a polynomial equation with a finite number of roots, whereas that for a differential-difference equation possesses an infinite number of roots. This infinity of roots permits the understanding of parasitic oscillatory phenomena which are alien to an unretarded theory. Several classes of linear differential-difference equations are discussed in detail. It is observed experimentally that the parasitic oscillations arise spontaneously and exist with stable amplitudes. This is a well-known characteristic of nonlinear systems. Consequently it is plausible to explain the observed results on the basis of a nonlinear differential-difference equation. The formal methods of nonlinear mechanics are now applied to derive the solutions corresponding to these oscillations. There is experimental confirmation of the results obtained in this manner.

R. Bellman (Stanford University, Calif.).

Myškis, A. D. Hystero-differential equations. Uspehi Matem. Nauk (N.S.) 4, no. 1(29), 190–193 (1949). (Russian)

The author is interested in a type of functional equation of which a simple example is $y''(x)+f(x)y(x-g(x))=0$. In this note he outlines some results concerning the oscillation of the solutions. No proofs are given.

R. Bellman (Stanford University, Calif.).

Myškis, A. D. General theory of differential equations with retarded arguments. Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 99–141 (1949). (Russian)

An extensive bibliography is given, as well as an outline of the past developments and of the current state of the subject. With a few notable exceptions, most of the past literature dealt with rather special problems and often formally and with inadequate rigor. The major part of this work embodies contributions by the author to the general theory. A considerable part of past study had been with respect to differential-difference equations

$$(1)\qquad \sum_{l=0}^{m}\sum_{k=1}^{k_l} a_k{}^l y^{(l)}(x+h_k{}^l)=0$$

($a_k{}^l$, $h_k{}^l$ constants; $h_k{}^l$ real). Euler's method consists in substituting in (1) $y=e^{\lambda x}$, giving for λ a transcendental equation (2) $E(\lambda)=0$; a formal solution of (1) is $\sum C_s e^{\lambda_s x}$ (λ_s are the roots of (2)). The author studies the equation

$$(3)\qquad y^{(m)}(x)=f(x,\cdots,y^{(l)}(x-\Delta_k{}^l(x)),\cdots),$$

where $m>0$ and the arguments of f are x and the displayed derivatives for $l=0,\cdots,m-1$ and $k=1,\cdots,k_l$ ($k_l\geqq 0$); all the "retards" $\Delta_k{}^l(x)\geqq 0$ (in physical problems, involving equation (3), there is a good reason for this). A generalization of (3) is the system

$$(4)\qquad y_i{}^{(m_i)}(x)=f_i(x,\cdots,y_j{}^{(l)}(x-\Delta_{jk}{}^l(x),\cdots)$$

($i=1,\cdots,n$), where $j\leqq n$, $l\leqq m_j-1$, $k\leqq k_{jl}$ ($k_{jl}\geqq 0$; $\Delta_{\cdots}^{\cdots}(x)\geqq 0$). The f_i are assigned in a region G of the K-dimensional space (K=number of arguments in the f_i); the "retards" are assigned on a finite segment $[A, B]$.

A precise formulation (involving certain functions $\varphi_{jk}{}^l(x)$) of the initial problem, appropriate to (3) or (4), is given; this is an analogue to the Cauchy problem for differential equations. Here are some of the author's further results. (I) Let the f_i, $\Delta_{\cdots}^{\cdots}(x)$, $\varphi_{\cdots}^{\cdots}(x)$ be continuous, the $y_j{}^l=\varphi_{jk}{}^l(A)$ arbitrary, and let the point $\{A,\cdots,\varphi_{jk}{}^l(A-\Delta_{jk}{}^l(A)),\cdots\}$ be in G; then on some subsegment $[A, B_1]$ of $[A, B]$ (4) has a solution satisfying the initial conditions. Another result (II) refers to the problem of continuation of a solution when the conditions of (I) hold. (III) Under quite general conditions the solution $\{y_j(x)\}$ involved in (I) is unique. The concept of a contingent (generalizing the notion of a tangent) is used with good effect. A differential equation in contingents is introduced, with retarded arguments. It is shown that under suitable continuity conditions it has a solution; a study of continuations of the latter is made. The author then proceeds under the assumptions securing existence of a solution and shows that the "integral funnel" W for $A\leqq x\leqq B_1$ [the set of all points of all solutions satisfying the given initial conditions] is a compact connected set and that the intersection of W with any plane $x=C$ ($A\leqq C\leqq B_1$) is connected. Finally, a linear nonhomogeneous system of differential equations with retarded arguments is considered and upper bounds for the absolute values of the constituent elements of its solution are found. *W. J. Trjitzinsky.*

Revuz, André. Sur l'équation fonctionnelle $f[f(x)]=x$. Bull. Tech. Univ. Istanbul 1, 29–35 (1948). (French. Turkish summary)

It is proved that, if $f(x)$ satisfies the conditions (I): $xf(x)>0$, $f(0)=0$, $f[f(x)]=x$, and if also $f(x)$ possesses the Darboux property, then (II) $f(x)\equiv x$. A construction is given for functions satisfying (I) but not (II).

A. G. Walker (Sheffield).

Pellegrino, Franco. Su alcune equazioni funzionali. Boll. Un. Mat. Ital. (3) 4, 135–139 (1949).

The functional equation (1) $v(x)f(1/x)=f(x)$ can be solved for f only if (2) $v(1/x)=1/v(x)$; thus $v(x)=q(\log x)$ ($q(t)$ is an odd function). The author gives two (equivalent) forms of the general solution f of (1):

$$f(x)=p(\log x)v(x)\log v(x)/(v(x)-1)$$

and

$$(1')\qquad f(x)=p(\log x)v(x)/[v(x)+1]$$

[$f(x)=p(\log x)[v(x)]^{\frac{1}{2}}$ would be a shorter form of the general solution], where $p(t)$ is an even function. The author makes use of the fact that the general solution of (3) $v(x)f[u(x)]=f(x)$ is (4) $f(x)=f_1(x)f^*(x)$, where $f_1(x)$ is the general solution of $f_1[u(x)]=f_1(x)$ and $f^*(x)$ is an arbitrary particular solution of (3). For (1) this gives $f_1(x)=p(\log x)$ ($p(t)$ is even) and

$$f^*(x)=v(x)\log v(x)/[v(x)-1]$$

or

$$f^*(x)=v(x)/[v(x)+1]$$

[$\sqrt{v(x)}$ would be still shorter]; the particular solutions of (1) are found by the author really by trial. [A more systematic way which also does not use (4) and leads to the result that the general solution of (1) has e.g. the form (1'), and similarly the two other forms given above, would be: (1) implies $f(1/x)=v(1/x)f(x)$, thus by addition $g(x)=f(x)[1+v(1/x)]=f(1/x)[1+v(x)]=g(1/x)$ and this gives $g(x)=p(\log x)f(x)=p(\log x)v(x)/[v(x)+1]$, q.e.d.] The author also shows that the functional equation $u(x)f[u(x)]=f(x)$, where $u(x)$ is involutory, i.e., $u[u(x)]\equiv x$, can be solved only if $u(x)=1/x$ and thus its general solution has the form $f(x)=p(\log x)/(1+x)$, or

$$f(x)=p(\log x)\log x/(x-1),$$

or $f(x)=p(\log x)x^{-\frac{1}{2}}$. *J. Aczél* (Szeged).

Integral Equations

Bellman, Richard. A note on the summability of formal solutions of linear integral equations. Duke Math. J. 17, 53–55 (1950).

For the integral equation

$$(1)\qquad u(s)=f(s)+\lambda\int_0^1 K(s,t)u(t)dt$$

with a continuous kernel the author studies the summability of the formal Liouville series for (1) by a regular method of summability S. [Summability of this series by Euler-Knopp methods has been considered by H. Bückner, same J. 15, 197–206 (1948); these Rev. 9, 624.] He shows that the Liouville series is S summable to the Fredholm solution of (1) for all noncharacteristic λ, if the set R where S sums the series $\sum z^n$ to $(1-z)^{-1}$ is the whole plane except $z=1$. A similar result holds for methods S with a smaller set R. *G. G. Lorentz* (Toronto, Ont.).

★**Heins, Albert E. Systems of Wiener-Hopf integral equations and their application to some boundary value problems in electromagnetic theory.** Proc. Symposia Appl. Math. 2, 76–81 (1950). $3.00.

This paper explains clearly but without detail how certain non-trivial generalizations of Sommerfeld's classical diffraction problem depend on the solution of systems of Wiener-Hopf integral equations

$$\sum_{j=1}^{n}\int_0^\infty K_{ij}(x-x')E_i(x')dx'+F_j(x)=0, \qquad j=1,2,\cdots,n,$$

for $x>0$, the functions $F_j(x)$ being defined only for $x>0$. Just as in the case $n=1$, the system can be solved by means of complex Fourier transforms; and a solution is given by means of matrix theory. If $k(w)$ is the $n\times n$ matrix of the Fourier transforms of the kernels $K_{ij}(x)$, it is first necessary to express $k(w)$ in the form $e^{l(w)}$, where $l(w)$ is regular in a strip of the w-plane; and this involves a knowledge of the characteristic values (latent roots) of $k(w)$. Next $l(w)$ must be represented as a sum of commuting matrices with overlapping half-planes of regularity. This restricts the range of application of the theory; for it is pointed out that the method is inapplicable to the problem of radiation from a semi-infinite coaxial line. *E. T. Copson* (Dundee).

Pollard, Harry. Note on the kernel $\phi(-|x-y|)$. Quart. Appl. Math. 7, 473 (1950).

The author notes that a theorem of H. P. Thielman [same Quart. 6, 443–448 (1949); these Rev. 10, 459] is incorrect and replaces it by a theorem of similar scope which is correct. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Parodi, Maurice, et Poli, Louis. Résolution d'équations intégrales par transformation en équations à noyaux réciproques. C. R. Acad. Sci. Paris 230, 37–40 (1950).

If the Laplace transforms of $g(t)$ and $K(t,x)$ are respectively $\theta(s)$ and $N(s,x)$, then the integral equation

$$(1)\qquad \int_0^\infty K(t,x)f(x)dx=g(t)$$

can be transformed into

$$(2)\qquad \int_0^\infty N(s,x)f(x)dx=\theta(s).$$

In some cases it is possible to determine the unknown function $f(x)$ from (2) more readily than from (1). This is illustrated by examples in which N is a reciprocal kernel, in particular the kernel of the Hankel transformation or of the Fourier transformation. Instead of the Laplace transformation, the inverse Laplace transformation can be applied to (1) with similar results. *A. Erdélyi.*

Magnaradze, L. G. On a general theorem of I. I. Privalov and its applications to certain linear boundary problems of the theory of functions and to singular integral equations. Doklady Akad. Nauk SSSR (N.S.) 68, 657–660 (1949). (Russian)

Let L be a simple closed piecewise continuous curve, $t=x(s)+iy(s)$ $(0\leqq s\leqq l;\ s$, length of curve). With φ assigned on L, consider (1) $\psi(t_0)=(2\pi i)^{-1}\int_L(t-t_0)^{-1}\varphi(t)dt$ (t_0 on L). When L is a circle, one considers

$$(2)\qquad g(x_0)=(2\pi)^{-1}\int_0^{2\pi}f(x)\cot\tfrac{1}{2}(x_0-x)dx,$$

$f(0)=f(2\pi)$, $0\leqq x_0\leqq 2\pi$. Let

$$\omega(\tau;\varphi)=\sup_{|s_2-s_1|\leqq\tau}|\varphi(t(s_2))-\varphi(t(s_1))|.$$

One has

$$(3)\qquad C\omega(\tau_0;\psi)\leqq\omega(\tau_0;\varphi)+\int_0^{\tau_0}\omega(\tau;\varphi)\tau^{-1}d\tau+\tau_0\int_{\tau_0}^{l}\omega(\tau;\varphi)\tau^{-2}d\tau$$

($0<\tau_0\leqq$small l_0), $C=$constant>0. This is analogous to a result, relating to (2), due to A. Zygmund. Let $H_\alpha\Lambda_p$ be the class of functions F for which

$$(4)\qquad \omega(\tau;F)=O(\tau^\alpha\log^{-p}(1/\tau)\cdot\log_2^{-p_2}(1/\tau)\cdot\cdots\cdot\log_m^{-p_m}(1/\tau))$$

($0<\tau\leqq l_0$), with $\alpha, p, p_2, \cdots, p_m$ real. If φ is in $H_\alpha\Lambda_p$, then ψ is in a certain $H_{\alpha_0}\Lambda_{p_0}$. Assume now that $p_2=\cdots=p_m=0$; for $H_0\Lambda_p$ write Λ_p; designate $H_0\Lambda_p$ by D_p, when the symbol O in (4) is replaced by o. Let I_p $(p\geqq 0)$ be the class of F for which $\tau^{-1}\omega(\tau;F)(\log(1/\tau))^p$ is integrable on $[0,l]$. If φ is in I_{p+1}, ψ is in I_p; $\Lambda_p\subset I_0$ (all $p>1$), $I_{p+1}\subset\Lambda_{p+2}\subset I_p$, $I_p\subset D_{p+1}\subset\Lambda_{p+1}$. Let $\Lambda_\infty=\prod\Lambda_p$, $D_\infty=\prod D_p$, $I_\infty=\prod I_p$; then $\Lambda_\infty=D_\infty=I_\infty$. The latter class is more general than the Hölder class. Applications are made (when L is a simple closed piecewise continuous curve) to related Riemann-Hilbert boundary value problems and to singular integral equations with Cauchy kernels. Such extensions to classes of functions more general than Hölder classes are possible largely with the aid of certain results due to I. I. Privalov [Bull. Soc. Math. France 44, 100–103 (1916); Doklady Akad. Nauk SSSR (N.S.) 23, 859–862 (1939); these Rev. 1, 305]. *W. J. Trjitzinsky* (Urbana, Ill.).

Dubrovskiĭ, V. M. Systems of nonlinear integral equations. Uspehi Matem. Nauk (N.S.) 4, no. 2(30), 176–177 (1949). (Russian)

The author states the following result. Consider a system $u_i(x)=f_i(x)+\lambda\int K_i[x,y,u_1(y),\cdots,u_n(y)]dy$, $i=1,\cdots,n$, integration over a closed set E, $f_i(x)$ continuous; write $K_i(x,y,u_1,\cdots,u_n)=K_i(x,y,u)$;

$$|K_i(x,y,u)|\leqq A_{0,i}(x,y)+\sum_1^n A_{\nu,i}(x,y)|u_\nu|^{\alpha_\nu}$$

(x, y in E; $0<\alpha_\nu<1$; $A_{\nu,i}\geqq 0$, summable); let the supremum of $\int A_{\nu,i}(x,y)dy$ for x in E be finite; let

$$|K_i(x,y,u)-K_i(x,y,u')|\leqq\sum_1^n B_{\nu,i}(x,y)|u_\nu-u_\nu'|^{\beta_\nu},$$

where $B_{\nu,i}\geqq 0$ and sup (x in E)$\int B_{\nu,i}(x,y)dy$ is finite;

$$|K_i(x_1,y,u)-K_i(x_2,y,u)|\leqq|x_1-x_2|^{\gamma_i}C_i(y,u_1,\cdots),$$

with $0<\gamma_i\leqq 1$ and sup $C_i=F_i(y,N)$ (the symbol sup corresponds here to $u_1,\cdots,u_n$ of absolute value not exceeding N) summable for every $N>0$. Under the above conditions there exist functions $u_j(x)$, continuous in E, satisfying the system for all λ. *W. J. Trjitzinsky* (Urbana, Ill.).

Vaĭnberg, M. M. On the characteristic values of some systems of nonlinear integral equations. Uspehi Matem. Nauk (N.S.) 4, no. 3(31), 130–132 (1949). (Russian)

Under consideration is the system

$$(1)\qquad \mu_iu_i(x)=\int K_i(x,y)g_i(u(y),y)dy$$

($i=1,\cdots,n$); $g_i(u,y)=g_i(u_1,\cdots,u_n,y)$; the $K_i(x,y)$ symmetric, positive (in the sense of integral equations); $g_i(u,x)$ continuous in $u=(u_1,\cdots,u_n)$ for every x in B; (2) $g_i(u,x)=(\partial/\partial u_i)G(u,x)$, (3) $g_i(0,x)=G(0,x)=0$ (for x in B); integration over a bounded region B. The characteristic element of (1) is the set of functions $\psi(x)=(\psi_1(x),\cdots,\psi_n(x))$, satisfying (1) for some μ_i and having its norm (in the sense of the space $L_{2,n}$) distinct from zero. The following results

are stated. The system (1) has at least a denumerable infinity of characteristic elements, approaching 0, if the operator $hu=(h_1u, \cdots, h_nu)$ (with $h_iu=g_i(u(x), x))$ is continuous (that is, $\|u-u_n\|\to 0$ implies $\|hu-hu_n\|\to 0$) in the sphere $\|u\|\leqq c$, while (4) $\int\int K_i^2(x, y)dxdy$ $(>0)<\infty$. The system has at least a denumerable infinity of characteristic elements, approaching 0 (in norm), if the g_i satisfy $|g_i(u, x)|\leqq a_i(x)+b_i\sum|u_k|+c_i(x)\sum|u_k|^{\frac{1}{2}}$, where $0\leqq a_i(x)$ in L_2, $0\leqq b_i=$constant, $c_i(x)$ in L_2. There exists a sequence of positive characteristic values, approaching 0, provided: (I) the $K_i(x, y)$ are symmetric, positive and satisfy (4); (II) the $g_i(u, x)$ are continuous in u, measurable in x and $|g_i(u, x)|\leqq a_i(x)+b_i\sum|u_k|^{p_{ik}}$, with $a_i(x)$ in L_2, $b_i=$constant and $0<p_{ik}<1$; (III) sgn $g_i(u, x)=$sgn u_i.

W. J. Trjitzinsky (Urbana, Ill.).

Dolph, C. L. Nonlinear integral equations of the Hammerstein type. Trans. Amer. Math. Soc. 66, 289–307 (1949).

Gegenstand der Untersuchung ist die nichtlineare Integralgleichung von der Form

$$\psi(x)=\int_a^b K(x, y)f[y, \psi(y)]dy.$$

Dabei ist $K(x, y)$ ein stetiger Kern und $f(x, y)$ eine nichtlineare Funktion von y, von der Stetigkeit im Intervall $a\leqq x\leqq b$ und $-\infty<y<\infty$ vorausgesetzt wird. Für die aufzustellenden Existenz- und Eindeutigkeitstheoreme werden insbesondere weiterhin folgende einschränkende Ungleichungen in Diskussion gezogen:

$$(\mathrm{A}^1)\qquad \mu_N y^2-C_N\leqq 2\int_0^y f(x, y)dy\leqq \mu_{N+1}y^2+C_{N+1};$$

$$(\mathrm{B}^1)\qquad \begin{aligned} &\mu_N y-A\leqq f(x, y)\leqq \mu_{N+1}y+A, && y>y_0,\\ &\mu_{N+1}y-A\leqq f(x, y)\leqq \mu_N y+A, && y<y_0;\end{aligned}$$

$$(\mathrm{C}^1)\qquad \lambda_N<\mu_N\leqq\frac{f(x, y_2)-f(x, y_1)}{y_2-y_1}\leqq\mu_{N+1}<\lambda_{N+1}.$$

Dabei sind C_N, C_{N+1}, A, μ_N, μ_{N+1} Konstante, wobei $A>0$, $\lambda_N<\mu_N<\mu_{N+1}<\lambda_{N+1}$ ist. Die λ_ν sind die Eigenwerte für die entsprechende lineare Integralgleichung. Der Fortschritt gegenüber früheren Arbeiten [Hammerstein, Acta Math. 54, 117–176 (1930); R. Iglisch, Math. Ann. 108, 161–189 (1933); Monatsh. Math. Phys. 42, 7–36 (1935); M. Golomb, Math. Z. 39, 45–75 (1934)] liegt insbesondere darin, dass in den genannten Arbeiten die entsprechenden Fragen nur unter Voraussetzungen von analogen Ungleichungen diskutiert wurden, wo an Stelle des Intervalls zwischen zwei beliebigen Eigenwerten das Intervall unterhalb des ersten Eigenwertes figurierte.

Im ersten Teil der Arbeit zeigt der Verfasser, dass Voraussetzung B^1 genügt, um a priori eine Schranke für die Norm aller möglichen Lösungen anzugeben. Dies genügt bereits, um nach der Methode von Leray und Schauder [Ann. Sci. École Norm. Sup. (3) 51, 45–78 (1934)] die Existenz einer Lösung zu beweisen. Ferner zeigt der Verfasser, dass die Annahme C^1 genügt, um im Anschluss an die genannte Arbeit von Golomb einen konstruktiven Existenzbeweis nach der Methode der fortschreitenden Annäherungen zu liefern und die Eindeutigkeit der Lösungen zu erweisen. Bei diesen Annahmen entspricht der Kern einem vollstetigen Hermitischen Operator, wobei auch negative Eigenwerte zulässig sind.

Im zweiten Teil der Arbeit zeigt der Verfasser zunächst durch eine einfache Betrachtung im dreidimensionalen Raum, dass unter der Annahme A^1 vom Standpunkt der Topologie aus die Existenz einer Schranke für die Lösung a priori nicht als gesichert erscheint. Gerade dieser Umstand veranlasst nun den Verfasser, mit Hilfe der Methoden der Variationsrechnung im grossen einen Existenzbeweis zu führen, der nicht eine a priori-Schätzung für die Norm einer Lösung nötig macht. Freilich wird hier der Beweis nur unter engeren Voraussetzungen durchgeführt und auch die Bedingung A^1 wird durch eine wesentlich engere Bedingung vom gleichen Typus ersetzt.

Im dritten Teil der Arbeit bespricht der Verfasser Anwendungen, u.z. zunächst das Randwertproblem der nichtlinearen Differentialgleichungen vom Typus $L(u)=f(x, y, u)$, $u=\gamma$ längs des Randes, wobei

$$L(v)=\frac{\partial}{\partial x}\left(p\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial v}{\partial y}\right)$$

bedeutet. Dabei wird vorausgesetzt, die zu $L(u)=0$ gehörige Greensche Funktion liefere einen vollstetigen Operator im Hilbertschen Raum. Ferner wird bezüglich der Funktion $f(x, y, u+v)$ eine Bedingung von der Form B^1 bzw. C^1 vorausgesetzt. Schliesslich verweist der Verfasser auf ein u.a. von Hammerstein behandeltes verallgemeinertes Schwingungsproblem von Duffing. *P. Funk* (Wien).

Ščelkunov, V. A. On an integral equation with Riemann-Stieltjes integrals. Doklady Akad. Nauk SSSR (N.S.) 69, 137–139 (1949). (Russian)

Let (E) be the segment $[a, b]$ and (ω) any semisegment $[\alpha, \beta]\subset[a, b]$ for $\beta\neq b$ and the segment $[\alpha, b]$, when $\beta=b$; $\omega=$length of (ω). Let $K(t, \omega, x, y)=K(t, x, y)f(x, \omega)$, where K is continuous in R $(a\leqq t, x, y\leqq b)$ and $f(x, \omega)$ is the mean additive function of the domain (ω), defined by the relations $f(x, \omega)=0$ (x not in (ω)), $=\omega^{-1}$ (x in (ω)). Under consideration is the equation

$$(1)\qquad u(x, y)=\varphi(x, y)+\lambda\int_a^b\int_{(E)}K(t, \omega, x, y)u(t, \xi)dtd\omega,$$

which is expressible as

$$(2)\qquad u(x, y)=\varphi(x, y)+\lambda\int_a^b K(t, x, y)u(t, x)dt;$$

here $u(t, x)$ is continuous in I $(a\leqq t, x\leqq b)$, ξ is in (ω). Equation (2) is of importance in the theory of probability and has been studied by V. Romanovsky [Acta Math. 59, 99–208 (1932)], who used the Fredholm idea of replacing an integral equation by a system of algebraic equations. The author shows the possibility of treating (1) directly, using N. Gunther's theory of integral equations whose kernels are mean additive functions of domains; one thus obtains the classic Fredholm formulas, with Riemann integrals replaced by suitable Riemann-Stieltjes integrals. The author accordingly formulates for (1) the three fundamental theorems of Fredholm. *W. J. Trjitzinsky* (Urbana, Ill.).

***Hattan, Corinne Rose. Lebesgue-Stieltjes Integral Equations of the First Kind.** Abstract of a Thesis, University of Illinois, 1946. 9 pp.

Let $u(e_x)$ be a completely additive continuous nonnegative measure-function, which exists on Borel sets $(e_x)\subset(D_x)$, and let $F(e_x)$ be a continuous additive function, having a derivative $dF(e_x)=f(x)$ with respect to the measure function u. Trjitzinsky [Acta Math. 74, 197–310 (1941); these Rev. 7, 304] has discussed solutions of the equation

$$(1)\qquad \int_{(D_x)}L(x, y)\phi(x)du(e_x)=f(y),$$

$\phi(x)$ unknown. Using similar methods the author treats the equation

$$\int_{(D_x)}\left[\int_{(e_y)} L(x, y)du(e_y)\right]d\psi(e_x)=F(e_y), \tag{2}$$

$\psi(e_x)$ unknown. When $L(x, y)$ and $F(e_x)$ satisfy certain integrability conditions, a spectral representation of a solution of (2) is given. A solution of (2) is shown to give rise to a solution of (1), and a solution of (1) gives rise to solutions of (2) that differ at most by a singular function.

F. G. Dressel (Durham, N. C.).

Germay, R. H. Sur des équations intégro-différentielles récurrentes. Bull. Soc. Roy. Sci. Liége **18**, 250–258 (1949).

For a system of "recurrent" integro-differential equations of the form

$$y_n'(x)=F_n\Big[x, y_n(x), y_{n+1}(x);$$
$$\int_{x_0}^{x} f_{n,1}\{x, s; y_n(s), y_{n+1}(s)\}ds, \cdots,$$
$$\int_{x_0}^{x} f_{n,p}\{x, s; y_n(s), y_{n+1}(s)\}ds\Big], \quad n=1, 2, \cdots,$$

the author applies the method of successive approximations to show that under suitable conditions on the functions there is an interval $x_0 \leqq x \leqq x_0+h$ on which there exists a unique solution satisfying $y_n(x_0)=y_0$ $(n=1, 2, \cdots)$.

W. T. Reid (Evanston, Ill.).

Goldoni, Gino. Teorema di unicità per una equazione integro-differenziale che regge il fenomeno di diffusione dei "neutroni termici" in paraffina. Atti Sem. Mat. Fis. Univ. Modena **3**, 138–141 (1949).

The author shows that the method recently proposed by Pignedoli [same journal **2**, 96–107 (1948); these Rev. **10**, 716] for solving the time dependent equation of transfer for a linear law of scattering leads to the unique solution of the problem. *S. Chandrasekhar* (Williams Bay, Wis.).

Drăganu, Mircea. Sur la correction relativiste dans quelques problèmes de diffusion. J. Phys. Radium (8) **10**, 301–304 (1949).

The equation governing the diffusion of neutrons is generalized to allow for the relativistic mass variation with velocity. *S. Chandrasekhar* (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Shimoda, Isae, and Iseki, Kiyosi. General analysis in abstract spaces. I. J. Osaka Inst. Sci. Tech. Part I. **1**, 61–66 (1949).

The first part of this paper deals with generalizations to linear topological spaces of some well-known theorems on operators in Banach spaces (theorems on the continuity and mapping of bounded sets into bounded sets by "polynomial" operators, and theorem 5, p. 80 of Banach's "Théorie des Opérations Linéaires" [Warszawa, 1932], together with a familiar corollary). The authors use some, but not all (1, 3, 4, 5), of the axioms of von Neumann [Trans. Amer. Math. Soc. **37**, 1–20 (1935), p. 4] for a linear topological space. The axioms selected are not by themselves sufficient to define a linear topological space. In particular, a single point need not be a closed set. The authors have also omitted the axiom (no. 6) used by von Neumann to prove that scalar multiplication is continuous, though such continuity is needed in their proofs. A second oversight occurs in the definition of homogeneous polynomials. Homogeneity alone is not all that is usually required. The authors' definition would make $(x^3+y^3)^{\frac{1}{3}}$ a homogeneous polynomial of first degree.

The second part of the paper contains the following theorem (where E_1, E_2, E_3 are complex Banach spaces, and $E=E_1\times E_2$). If $f(x_1, x_2)$ on E to E_3 is analytic on the boundary of a bounded domain Δ in E, then it is analytic in Δ. This is an inexactly worded extension of a classical theorem of Hartogs [W. F. Osgood, Lehrbuch der Funktionentheorie, v. 2, part 1, Teubner, Leipzig, 1924, p. 206]. The boundary of Δ should be assumed connected, and the conclusion is a statement about the possibility of analytic continuation. In the opinion of the reviewer there are serious gaps in the proof. *A. E. Taylor* (Los Angeles, Calif.).

Everett, C. J., and Ryser, H. J. Rational vector spaces. I. Duke Math. J. **16**, 553–570 (1949).

A detailed study is made of vector spaces over the field of rationals. First, some results on cardinal numbers of such spaces are obtained, culminating in the theorem that every two infinite-dimensional rational vector spaces of the same cardinal number are isomorphic. Here and elsewhere in the paper, the dominating idea is that of a Hamel basis of a rational vector space, together with certain spaces of rationally-valued functions. Next, a rationally-valued inner product is postulated, which permits the introduction of orthogonal bases, the Schmidt orthogonalization process, etc., for such "rational inner product spaces." In the finite-dimensional case, to which most of the remainder of the paper is devoted, the study of "equivalence" of two rational inner product spaces reduces to a study of the Minkowski-Hasse invariants of associated quadratic forms under rational transformations. Not all n-dimensional rational inner product spaces are equivalent, but each is equivalent to a space consisting of all polynomials over the rationals of degree not exceeding n with an inner product of the form $(f, g)=\int_0^\infty f(x)g(x)d\psi$. This result is proved by use of a classical theorem on moments of Stieltjes. The Minkowski-Hasse invariants of the direct sum and direct product of two finite-dimensional rational inner product spaces are computed in terms of their values for the spaces themselves. The paper closes with some theorems on infinite-dimensional inner product spaces which are either denumerable or separable. In particular it is shown that every infinite-dimensional denumerable space has an orthonormal basis, in contrast to the situation for a finite number of dimensions.

D. H. Hyers (Los Angeles, Calif.).

Nachbin, Leopoldo. On strictly minimal topological division rings. Bull. Amer. Math. Soc. **55**, 1128–1136 (1949).

L'auteur dit qu'un corps topologique K (commutatif ou non) est minimal s'il n'existe pas de topologie strictement moins fine que la topologie de K, et compatible avec la structure de K; il dit que K est strictement minimal si pour tout espace vectoriel topologique $D=Ka$ de dimension 1 sur K, l'application $\lambda\rightarrow\lambda a$ de K sur D est un homéomorphisme; tout corps strictement minimal est minimal, tout corps valué et non discret est strictement minimal. L'auteur

étend aux espaces vectoriels topologiques sur un corps strictement minimal les propriétés bien connues des espaces normés sur un corps valué; en particulier il montre que si K est strictement minimal et complet, tous les espaces vectoriels topologiques de dimension n sur K sont isomorphes à K^n. *J. Dieudonné* (Nancy).

Nachbin, Leopoldo. A theorem of the Hahn-Banach type for linear transformations. Trans. Amer. Math. Soc. 68, 28–46 (1950).

A normed space $\mathfrak{E}$ is said to have the extension property if, given any normed space $\mathfrak{X}$ and a linear subspace $\mathfrak{S}$ of it, every continuous linear operator f from $\mathfrak{S}$ to $\mathfrak{E}$ has at least one continuous linear extension F from $\mathfrak{X}$ to $\mathfrak{E}$ having the same norm (or bound) as f. A collection $\mathfrak{C}$ of sets is said to have the binary intersection property if every subcollection of $\mathfrak{C}$, any two members of which intersect, has a nonempty intersection. It is shown that a real normed space has the extension property if and only if the collection of its closed spheres has the binary intersection property. The remainder of the paper is devoted to the consideration of spaces with these properties. In particular, it follows from results proved that the spaces (m) and (M) [S. Banach, Théorie des Opérations Linéaires, Warsaw, 1932; pp. 10, 11] have the extension property. [Note: the author uses "isomorphic" to mean what Banach [op. cit., p. 180] calls "equivalent."] *A. F. Ruston* (London).

✶Goodner, Dwight Benjamin. Projections in Normed Linear Spaces. Abstract of a Thesis, University of Illinois, 1949. i+3 pp.

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space Y containing X there is a projection T of norm less than or equal to s of Y onto X. A new proof is given of Akilov's theorem [Doklady Akad. Nauk SSSR (N.S.) 57, 643–646 (1947); 59, 417–418 (1948); these Rev. 9, 241, 358] that if X is of type B_1^+, if the unit sphere S of X has a least upper bound x_0, and if $\|x_0\| = s$, then X has property P_s. If X has property P_1 and p and q are distinct extreme points of S, then $\|p-q\| = 2$ and a line segment joining p to q lies wholly on the surface of S. In fact, any point on the surface of S may be joined to either p or $-p$ by a line segment lying entirely on the surface of S. If X has property P_1 and an extreme point on S, then X is equivalent to the space of all real-valued continuous functions on some disconnected compact Hausdorff space. Known examples of spaces having property P_1 are all abstract (M)-spaces with unit elements. *R. S. Phillips.*

Fullerton, R. E. Integral distances in Banach spaces. Bull. Amer. Math. Soc. 55, 901–905 (1949).

N. H. Anning and P. Erdös [same Bull. 51, 598–600 (1945); these Rev. 7, 164] and Erdös [ibid., 996 (1945); these Rev. 7, 164] showed that in a finite-dimensional Euclidean space there is a finite set, with any given number of points, such that (1) the distance between any two points in the set is an integer, and (2) the set is not contained in any straight line; and that no such infinite set can be found. This led the author to the following results. Let X be a Banach space which is strictly convex (i.e., the unit sphere does not contain any nonzero segment in its boundary). If an infinite set $A \subset X$ has property (1) and meets some straight line l in infinitely many points, then $A \subset l$. Moreover, if X fails to be strictly convex, there is some infinite set $A \subset X$ with properties (1) and (2). *L. Nachbin.*

Blumenthal, L. M., and Ellis, D. O. Notes on lattices. Duke Math. J. 16, 585–590 (1949).

Let L be a normed lattice, with norm $|x|$ and metric $d(x, y) = |x \cup y| - |x \cap y|$. The authors show that a necessary and sufficient condition for $b \epsilon L$ to be metrically between $a, c \epsilon L$, i.e., $d(a, c) = d(a, b) + d(b, c)$, is that $(a \cap b) \cup (b \cap c) = b = b \cup (a \cap c)$. Duality gives another equivalent condition. [Other such conditions were first given by V. Glivenko, Amer. J. Math. 58, 799–828 (1936).] Four points in a metric space are said to constitute a pseudo-linear quadruple if any three among them are metrically imbeddable in a line, but the quadruple is not. Two necessary and sufficient conditions for four distinct points of L to constitute a pseudo-linear quadruple are given. One of these applies to the distributive case only and reads as follows: the points can be so labelled that $a \cup c = b \cup d$ and $a \cap c = b \cap d$. Finally it is proved that L is metrically imbeddable in a Hilbert space if and only if L is simply ordered (in which case, of course, L is isomorphic to a normed sublattice of the line). *L. Nachbin* (Chicago, Ill.).

Tsuji, Masatsugu. On the integral representation of unitary and self-adjoint operators in Hilbert space. Jap. J. Math. 19, 287–297 (1948).

The spectral theorem for unitary operators U in Hilbert space is proved first and from it the spectral theorem for self-adjoint operators is deduced in the usual manner by considering the Cayley transform of U. The author remarks that he thus avoids proving the spectral theorem for bounded self-adjoint operators as an intermediary step and that the main idea of his proof is due to Wecken [Math. Ann. 110, 722–725 (1935)] except that the present proof uses a Fourier expansion instead of Wecken's power series. *F. I. Mautner* (Cambridge, Mass.).

See the note to p. 241, Tsuji.

Dixmier, Jacques. Étude sur les variétés et les opérateurs de Julia, avec quelques applications. Bull. Soc. Math. France 77, 11–101 (1949).

Let $\mathfrak{H}$ be a Hilbert space. Let $\mathfrak{L}$ be the smallest family of subspaces of $\mathfrak{H}$ which contains all closed subspaces and which contains $D \cap D'$ and $D + D'$ whenever it contains D and D'. Then $D \epsilon \mathfrak{L}$ if and only if it is the domain of some closed linear operator or equivalently if and only if it is the range of some bounded linear operator. Let $\mathfrak{R}$ be the smallest family of linear operators which contains the closed operators and with any two operators their sum and product. Then an operator is in $\mathfrak{R}$ if and only if its graph is in $\mathfrak{L}$. In this paper these theorems are proved and a detailed and systematic study is made of the classes of operators and subspaces thus defined. In particular, it turns out to be possible to find a complete set of unitary invariants for the members of $\mathfrak{L}$. Among the topics studied are: (a) the family of infinite dimensional closed subspaces contained in a member of $\mathfrak{L}$; (b) the family of operators in $\mathfrak{R}$ having given members of $\mathfrak{L}$ for domain and range; (c) the existence of families of mutually disjoint members of $\mathfrak{L}$ which are individually dense in $\mathfrak{H}$. Throughout an important role is played by the following classification of members of $\mathfrak{L}$: D is in class 1 if it is closed and infinite dimensional; D is in class 2 if it is not in class 1 but contains members of class 1; D is in class 3 if it contains no members of class 1. The contents of this paper have been well summarized in three notes [C. R. Acad. Sci. Paris 223, 971–972 (1946); 224, 180–181, 255–257 (1947); these Rev. 8, 518, 519]. In the reviews of these notes one will find more detailed statements of the results obtained under (a) and (b) as well

as the theorem on unitary invariants. A typical result under (c) asserts that if $D_1, D_2, \cdots$ is an arbitrary sequence of members of $\mathfrak{L}$ no one of which has finite deficiency then there exist unitary operators $U_2, U_3, \cdots$ such that $D_1, U_2(D_2), U_3(D_3), \cdots$ are mutually disjoint. Results of this sort have implications about the possible triviality of the domains of definition of sums and products of closed operators. Among others the author obtains a generalization of Neumark's result to the effect that there exists a closed symmetric operator whose square is defined only at 0.
G. W. Mackey (Princeton, N. J.).

Dixmier, J. Sur les variétés J d'un espace de Hilbert. J. Math. Pures Appl. (9) **28**, 321–358 (1949).

In this paper the author settles several questions left open in the paper reviewed above. The results are too numerous and complicated to be described with any degree of completeness in a brief review. The following are typical. (1) Let D be a member of $\mathfrak{L}$. If V_1 and V_2 are closed subspaces of $\mathfrak{H}$ and are subspaces of D write $V_1 \overset{D}{<} V_2$ whenever there exists a unitary operator U such that $U(D)=D$ and $U(V_1)\subseteq V_2$ and write $V_1 \overset{D}{=} V_2$ whenever $V_1 \overset{D}{<} V_2$ and $V_2 \overset{D}{<} V_1$. Then the ordering $\overset{D}{<}$ induces an ordering on the equivalence classes. Moreover, this ordering is complete and the resulting ordered set is isomorphic to one of six specified subsets of the rational numbers. (2) If D and D' are two members of $\mathfrak{L}$ which are dense in $\mathfrak{H}$ and not in class 3 then there exists a unitary transformation U such that $U(D)\subseteq D'$ only if the class of D is greater than or equal to that of D'. (The result when D and D' can be in class 3 is a little more complicated.) (3) Every lattice automorphism of $\mathfrak{L}$ is defined by a one-to-one bicontinuous map of $\mathfrak{H}$ on itself or by such a map combined with a "conjugation." (4) Every operator of type $A(V, V'; W, W'; L, L')$ (the definition of this concept is too involved to be presented here) is in $\mathfrak{R}$, is one-to-one and has range and domain which are dense and of class 1 or 2.
G. W. Mackey (Princeton, N. J.).

Dixmier, J. Les idéaux dans l'ensemble des variétés J d'un espace hilbertien. Ann. Fac. Sci. Univ. Toulouse (4) **10**, 91–114 (1949).

The author continues his investigation of the lattice $\mathfrak{L}$ of domains of closed linear operators in a Hilbert space [cf. the two preceding reviews]. An ideal in $\mathfrak{L}$ is a subset $\mathfrak{D}$ of $\mathfrak{L}$ such that if D and D' are in $\mathfrak{D}$ so is $D+D'$ and if $D\epsilon\mathfrak{D}$, $D'\epsilon\mathfrak{L}$ and $D\subset D'$ then $D'\epsilon\mathfrak{D}$. The ideal $\mathfrak{D}$ is invariant if whenever $D\epsilon\mathfrak{D}$ then every D' unitary equivalent to D is also in $\mathfrak{D}$. These ideals of subspaces are closely connected with ordinary algebraic ideals in the ring $\mathfrak{B}$ of all bounded linear operators in the relevant Hilbert space. Indeed if $\mathfrak{s}$ is an arbitrary two-sided ideal in $\mathfrak{B}$ then the ranges of the operators in $\mathfrak{s}$ form an invariant ideal in $\mathfrak{L}$. Conversely every invariant ideal $\mathfrak{D}$ in $\mathfrak{L}$ is obtainable in this manner from exactly one two-sided ideal in $\mathfrak{s}$; namely the ideal of all operators whose ranges are in $\mathfrak{D}$. In addition to establishing these results the author studies the invariant ideals in $\mathfrak{L}$ and interprets the results as theorems about the two-sided ideals in the ring $\mathfrak{B}$. In particular he obtains anew the Calkin–von Neumann results concerning the one-to-one correspondence between two-sided ideals in $\mathfrak{B}$ and "ideals of sequences" [J. W. Calkin, Ann. of Math. (2) **42**, 839–873 (1941); these Rev. **3**, 208]. The general situation here described is explored in some detail and numerous subsidiary results are presented. *G. W. Mackey* (Princeton, N. J.).

Dixmier, J. Les anneaux d'opérateurs de classe finie. Ann. Sci. École Norm. Sup. (3) **66**, 209–261 (1949).

Let H be a Hilbert space (not necessarily separable) and let M be a family of operators in H which is a ring in the sense of Murray and von Neumann. The author defines M to be of finite class if $A\epsilon M$ and $AA^*=I$ imply $A^*A=I$. His principal results are the following. Let M be of finite class and contain the identity I. Then there is a unique distributive mapping $A\rightarrow A^\natural$ of M onto its center such that (a) $A=A^\natural$ for all A in the center, (b) $(AB)^\natural=(BA)^\natural$ and (c) $A^\natural$ is self-adjoint and nonnegative whenever this is true of A. This mapping has (among others) the following additional properties: (1) $(AB)^\natural=AB^\natural$ for all A in the center; (2) if A is self-adjoint and nonnegative then $A^\natural=0$ implies $A=0$; (3) $A^{*\natural}=A^{\natural *}$; (4) it is continuous in the uniform but not in general in the weak and strong topologies; (5) if M is a factor it is continuous in the weak topology; (6) when restricted to the unit sphere it is continuous in the strong topology. Let K_A be the smallest convex uniformly closed subset of M which contains all operators UAU^{-1}, where A is a fixed member of M and U varies through the unitary members of M. Then K_A has just one element in common with the center of M and this element is A. It is this fact which is exploited in proving the existence of the mapping $A\rightarrow A^\natural$ [cf. the author's abstract, C. R. Acad. Sci. Paris **228**, 152–154 (1949); these Rev. **10**, 381]. When M is not of finite class the intersection in question is never empty but it contains more than one element for at least one A. The results concerning K_A hold also when the uniform topology is replaced by the weak topology. The operators A for which $A^\natural=0$ are precisely those in the uniform closure of the set of operators of the form $\sum_{i=1}^n\lambda_iU_iAU_i^{-1}$ where $\sum_{i=1}^n\lambda_i=0$, A is in M and each U_i is a unitary member of M.

When M is a factor $A^\natural$ is a multiple of the identity and this multiple is the Murray–von Neumann trace of A. Thus the author obtains a new and somewhat more direct proof of the existence of a trace for factors of finite class. His results go beyond those of Murray and von Neumann in that they apply to rings which need not be factors and to Hilbert spaces which need not be separable. He points out, however, that in the separable case many of them may be obtained by combining the Murray–von Neumann results on factors with the von Neumann decomposition of a general ring into factors and a number of supplementary considerations. *G. W. Mackey* (Princeton, N. J.).

Dixmier, Jacques. Sur les opérateurs self-adjoints d'un espace de Hilbert. C. R. Acad. Sci. Paris **230**, 267–269 (1950).

The author proves two theorems for which he says he has need in his work on rings of operators. (1) Let A be a nonnegative self-adjoint operator in the Hilbert space H. Let M be a closed subspace of H. Let $\mathfrak{L}$ be the family of all nonnegative self-adjoint operators in H which are bounded above by A, which are reduced by M and which are zero in $H\ominus M$. Then $\mathfrak{L}$ has a greatest element. (2) Let $\mathfrak{F}$ be a family of self-adjoint operators which is bounded above and which contains with any two operators a third which is greater than or equal to each. Then $\mathfrak{F}$ admits a least upper bound and this least upper bound is in the strong closure of $\mathfrak{F}$. *G. W. Mackey* (Princeton, N. J.).

Neĭmark, F. A. Extension of a Hermitian operator to one permutable with a given Hermitian operator. Doklady Akad. Nauk SSSR (N.S.) 66, 9–12 (1949). (Russian)

The author gives two solutions of the following problem. Given a bounded self-adjoint operator A in the Hilbert space $\mathfrak{H}$ with $\mathfrak{D}A=\mathfrak{H}$, a closed subspace $\mathfrak{M}$, and a bounded self-adjoint operator B with $\mathfrak{D}B=\mathfrak{M}$, $\mathfrak{R}B\subseteq\mathfrak{M}$, find conditions for B to be capable of a bounded self-adjoint extension B' on $\mathfrak{H}$ such that $AB'=B'A$. Both solutions use the following matrix-decomposition device. If $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ are mutually orthogonal closed subspaces spanning $\mathfrak{H}$, and if T is a linear operator on $\mathfrak{H}$ into $\mathfrak{H}$, put $T_{ik}=P_i(\mathfrak{H}_k|T)$, where P_i is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_i$ and $|$ means restriction to $\mathfrak{H}_k$ $(i, k=1, \cdots, n)$. We have $Tf=\sum_{i,k}T_{i,k}P_kf$ for $f\epsilon\mathfrak{H}$, and if $R=S\cdot T$, then $R_{ik}=\sum_\alpha S_{i\alpha}T_{\alpha k}$. If T is Hermitian, T_{ik} is Hermitian too in $\mathfrak{H}_k$, and $T_{ik}^*=T_{ki}$ [see F. J. Murray, Trans. Amer. Math. Soc. 37, 301–338 (1935)], which means $(T_{ik}f, g)=(f, T_{ki}g)$, $f\epsilon\mathfrak{H}_k$, $g\epsilon\mathfrak{H}_i$.

Method 1. The minimal closed subspace $\mathfrak{N}$ invariant for A and containing $\mathfrak{M}$ is the closure of the set E of all finite sums $\sum_kA^k\xi_k$, where $\xi_k\epsilon\mathfrak{M}$. This gives the following necessary and sufficient condition for the existence of B': $(A^kB\xi, \eta)=(A^k\xi, B\eta)$ $(\xi, \eta\epsilon\mathfrak{M}, k=1, 2, \cdots)$; $|\sum_kA^kB\xi_k|\leqq c|\sum_kA^k\xi_k|$ (finite sums, $\xi_k\epsilon\mathfrak{M}$, $c>0$). If we define, on E, the operator $B''\xi=\sum_kA^kB\xi_k$, where $\xi=\sum_kA^k\xi_k$, $\xi_k\epsilon\mathfrak{M}$, then B'' is well defined and can be extended, by continuity, to $\mathfrak{N}$. Putting

$$B'=\begin{Bmatrix}B'_{11} & B'_{12}\\ B'_{21} & B'_{22}\end{Bmatrix},$$

where $B'_{22}=B''$, B' is any bounded self-adjoint operator on $\mathfrak{H}-\mathfrak{N}$ permutable with $(\mathfrak{H}-\mathfrak{N})|A$ and $B'^*_{12}=B'_{21}$ we get all required B'.

Method 2. This relies on J. von Neumann's theorems [Ann. of Math. (2) 33, 294–310 (1932)] generalized by F. J. Murray [loc. cit.]. Putting $\mathfrak{H}_1=\mathfrak{M}$, $\mathfrak{H}_2=\mathfrak{H}-\mathfrak{M}$,

$$A=\begin{Bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{Bmatrix},\quad B'=\begin{Bmatrix}B, & 0\\ 0, & C\end{Bmatrix},$$

we obtain from the condition of permutability

$$(A_{12}A_{12}^*)B=B(A_{12}A_{12}^*).$$

Putting $H=(H_{12}H_{12}^*)^{\frac{1}{2}}$, we get $HB=BH$. We can write $A_{12}=HU$, where U carries the subspace $\mathfrak{N}_2$ where $A_{12}g=0$ into the 0 of $\mathfrak{H}_1$, and $\mathfrak{M}_2=\overline{\mathfrak{R}A_{12}^*}$ isometrically onto $\overline{\mathfrak{R}A_{12}}$. This leads to the equation $C=U^*BU$ on $\mathfrak{M}_2\subseteq\mathfrak{H}_2$. Thus, the existence of B' supposed, B' is determined on $\mathfrak{M}_2$. This procedure continued gives the invariant space of A on which B' is determined and, besides, furnishes some alternative necessary and sufficient conditions for the existence of B'.

O. M. Nikodým (Gambier, Ohio).

Brodskiĭ, M. S., and Livšic, M. S. On linear operational functions invariant with respect to the translation group. Doklady Akad. Nauk SSSR (N.S.) 68, 213–216 (1949). (Russian)

A proof of the following theorem is sketched. (a) If S is a hypermaximal symmetric operator on a Hilbert space H with a simple spectrum and with spectral family E_λ, and if $U(t)=\exp(itS)$ $(-\infty<t<\infty)$, then there exist linear operators A and B on H, satisfying the conditions (1) the domains D_A and D_{A^*} of A and A^* are dense in H; (2) $x\epsilon D_A$ implies $U(t)x\epsilon D_A$; (3) B is bounded and does not have 0 as a proper value; (4) $U(-t)AU(t)=A+tB$ for $-\infty<t<\infty$, if and only if for an arbitrary cyclic element x of S, the function $(E_\lambda x, x)$ is absolutely continuous and the set of λ's for which the derivative is nonzero differs from an open set by a set of measure zero. (b) Moreover, if all the conditions of (a) are satisfied, then there is a unitary transformation of H onto $L_2(M)$ for some open subset M of the reals, which takes S, $U(t)$, and B into multiplications by x, e^{itx}, and the bounded nonvanishing measurable function $b(x)$, respectively, and A and A^* are connected by the equation (i) $\bar{g}Af-f\overline{A^*g}=i^{-1}(d/dt)(bf\bar{g})$, for $f\epsilon D_A$ and $g\epsilon D_{A^*}$, $bf\bar{g}$ then differing from an absolutely continuous function on a set of measure zero. (c) Conversely, when A and A^* satisfy (1) and the other operators are as given in (b), then the conditions in (a) are satisfied, and almost every point of M is contained in a neighborhood in which A is a quasi-differential operator of first order.

I. E. Segal.

Fuglede, Bent. A commutativity theorem for normal operators. Proc. Nat. Acad. Sci. U. S. A. 36, 35–40 (1950).

There are two theorems in this note. (1) If B is a bounded, N a normal operator in Hilbert space, $N=\int zdE_z$ the canonical spectral resolution of N (N is not assumed bounded), and if B commutes with N $(BN\subseteq NB)$, then $BE_z=E_zB$ for all z and B commutes with "any" function of N, e.g., $BN^*\subseteq N^*B$ and $B^*N\subseteq NB^*$. An equivalent formulation of theorem 1 is: any normal operator N has the property P that the ring of all bounded operators B which commute with N is a self-adjoint ring. The author observes that the property P is characteristic of normal operators if only the bounded ones are considered. A counter-example for the general (unbounded) case is provided by theorem 2. There exists a closed (nonnormal) operator T, with domain dense, which does not commute with any bounded operator B which is not of the form cI (c, an arbitrary complex number; I, the identity operator). The operator T is defined in $L_2(-\infty, \infty)$ and is given by the formula: $Tf(x)=xf(x)+(d/dx)(f(x))$. The argument for theorem 2 involves an application of Liouville's theorem.

B. Gelbaum (Minneapolis, Minn.).

Julia, G. Sur la convergence uniforme simultanée des séries correspondant à deux opérateurs linéaires associés et totalement continus. Collectanea Math. 1, 61–66 (1948).

Let the pair A, A^* of adjoint operators in Hilbert space be given by $Ax=\sum a_k(x, e_k)$, $A^*x=\sum e_k(x, a_k)$. It is shown that a necessary and sufficient condition that A or A^* be totally continuous is that either one of these series be uniformly convergent on the unit sphere, and that this is equivalent to the condition that $\sum|(a_k, x)|^2$ should be uniformly convergent on the unit sphere. [Note: the notation for scalar products in the article is the opposite of that usually used and adopted in this review.]

J. L. B. Cooper (London).

★**Julia, Gaston. Quelques progrès récents dans la théorie des opérateurs linéaires de l'espace hilbertien.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 55–65. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

An expository lecture, sketching the results of a long series of [previously published and reviewed] notes of the author.

P. R. Halmos (Chicago, Ill.).

Frola, Eugenio. Algebre metrizzate di ordine infinito e operatori lineari negli spazi Hilbertiani. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 123–125 (1949).

The author announces that the bounded operators on Hilbert space form a topological ring in the uniform topology.

I. Kaplansky (Chicago, Ill.).

Landsberg, P. T. Notes on operators

$$F=\sum_{k=0}^{n}F_k(x)\left[\frac{d}{dx}\right]^k.$$

Math. Gaz. **33**, 113–115 (1949).

Conditions that operators of the above type be formally Hermitian, commutative and idempotent are found by direct calculation. Idempotency is shown impossible. In formula (4), $F^{*(i-u)}$ should read $F_i^{*(i-u)}$.

J. L. B. Cooper (London).

Povzner, A. On a class of Hilbert spaces of functions. Doklady Akad. Nauk SSSR (N.S.) **68**, 817–820 (1949). (Russian)

The author considers a Hilbert space H with inner product, denoted by (f, g), such that each f is a function on a fixed set M and, for each $z\epsilon M$, $\psi(z)=\sup_{\|f\|\leq 1}|f(z)|<\infty$. Then for each z there exists g_z such that $f(z)=(f, g_z)$ for all $f\epsilon H$ and $\psi(z)=\|g_z\|=(g_z(z))^{\frac{1}{2}}$. If $G(v, z)=g_z(v)$, then the function G is a Hermitian-positive kernel on M; conversely, every Hermitian-positive kernel in H can be represented as a G-function for such an M. Conditions involving determinants of the $G(a_i, a_j)$ are given for solutions of linear equations, and for a subset A of M to be a uniqueness set for the functions of H.

M. M. Day (Urbana, Ill.).

Abdelhay, José. Caractérisation de l'espace de Banach de toutes les suites de nombres réels tendant vers zéro. C. R. Acad. Sci. Paris **229**, 1111–1112 (1949).

The author states that a necessary and sufficient condition for a Banach space to be isomorphic [S. Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 180] to the space (c_0) of real sequences tending to zero is that it possess a base $\{w_i\}$ with the properties: (B_1) $\|w_i\|=1$ for $i=1, 2, \cdots$; (B_2) there is a constant K such that $\|w_1+\cdots+w_p\|<K$ for $p=1, 2, \cdots$; (J_1) $|f_i(x)|\geqq|f_i(y)|$ $(i=1, 2, \cdots)$ imply $\|x\|\geqq\|y\|$ for x and y in the given space; (J_2) for every element y of the space there is an element x such that $f_i(x)\geqq[f_i(y)]^+=\max[0, f_i(y)]$ $(i=1, 2, \cdots)$; where the linear functionals $\{f_i\}$ are defined by the identity $x=\sum_{i=1}^{\infty}f_i(x)w_i$. The author then considers the order relation defined by putting $x\geqq 0$ if and only if $f_i(x)\geqq 0$ for $i=1, 2, \cdots$. He finally states that under the above conditions $\|x\|^*=\sup_{1\leq i<\infty}|f_i(x)|$ is a norm for the given space, equivalent to $\|x\|$, and that, with this norm, the space is equivalent [S. Banach, loc. cit.] to (c_0). The paper contains no proofs.

A. F. Ruston (London).

Pellegrino, F. Die analytischen Funktionale und ihre Anwendungen. Mat. Tidsskr. B. **1949**, 31–62 (1949).

Expository paper.

Toeplitz, Otto. Die linearen vollkommenen Räume der Funktionentheorie. Comment. Math. Helv. **23**, 222–242 (1949).

Let π_r $(r\leqq\infty)$ be the class of complex sequences $x=\{x_n\}$ with $\limsup|x_n|^{1/n}\leqq 1/r$, and ρ_r $(r\geqq 0)$ the class of sequences with $\limsup|x_n|^{1/n}<1/r$. The author discusses these classes in the style and terminology of his earlier papers [Köthe and Toeplitz, J. Reine Angew. Math. **171**, 193–226 (1934); see also Köthe, Math. Ann. **114**, 99–125 (1937)]. The spaces π_r and $\rho_{1/r}$ are dual. The weak and strong topologies in each are studied in detail. Convergence of a sequence of points $x^{(k)}$ is connected with that of the analytic functions $f_k(z)=\sum_1^{\infty}x_n^{(k)}z^{n-1}$, and boundedness of a set of points with that of the corresponding set of functions. This gives an alternate approach to Vitali's theorem. An integral representation is obtained for the general linear transformation mapping π_r into $\pi_{r'}$, and ρ_r into $\rho_{r'}$, and the special case whereby a sequence x is mapped into its Faltung with a fixed sequence A, corresponding to multiplication of the corresponding functions, is studied. There seems to be a considerable overlap in content between part of this paper and recent papers by S. H. Doss [Proc. Math. Phys. Soc. Egypt **3** (1947), no. 3, 59–62 (1948); these Rev. **10**, 290] and Ganapathy Iyer [J. Indian Math. Soc. (N.S.) **12**, 13–30 (1948); these Rev. **10**, 380].

R. C. Buck.

Arens, Richard. Approximation in, and representation of, certain Banach algebras. Amer. J. Math. **71**, 763–790 (1949).

La première partie de ce mémoire contient des généralisations du théorème d'approximation de Stone; ce théorème lui-même est redémontré ici par des méthodes latticielles plus ou moins connues déjà; l'auteur démontre aussi le théorème suivant. Soient X un espace topologique, E un espace de Banach, $G=R\times E$ (R: nombres réels), A un sous-espace de l'espace des fonctions continues définies sur X et à valeurs dans G; supposons réalisées les conditions suivantes: (a) A est fermé pour la topologie de la convergence uniforme sur tout compact; (b) si $f, g\epsilon A$, alors A contient $f'\cdot g$ (f' est la "partie réelle" de f); (c) si $f=(f', f'')\epsilon A$, A contient la fonction $(\|f''\|^2, 0)$, ainsi que $(f', 0)$; alors pour que A contienne une fonction g donnée, il faut et il suffit que, quels que soient $x, y\epsilon X$ il existe $f\epsilon A$ avec $f(x)=g(x)$ et $f(y)=g(y)$. Comme conséquence de ce résultat, l'auteur prouve des théorèmes d'approximation pour les fonctions à valeurs dans une algèbre de matrices, et en particulier à valeurs dans le corps des quaternions.

La seconde partie traite d'une classe particulière d'algèbres normées; soit A une algèbre normée sur le corps réel, munie d'une involution * vérifiant les conditions suivantes: (a) ff^* est dans le centre de A pour tout $f\epsilon A$; (b) si f et g commutent, $\|f\|^2\leqq\|ff^*+gg^*\|$; dans ces hypothèses, soit X l'espace des homomorphismes de A dans le corps Q des quaternions, espace muni de la topologie faible habituelle; on a alors les propriétés suivantes: (1) X est un espace compact; (2) si à tout $f\epsilon A$ on associe, de façon évidente, une fonction $f(x)$ définie sur X et à valeurs dans Q, on a $\|f\|=\sup|f(x)|$, $f^*(x)=f(x)^*$; (3) soit Γ le groupe des automorphismes de Q considéré comme algèbre sur R; on sait que ces automorphismes sont tous intérieurs, et que Γ est isomorphe au groupe orthogonal à quatre variables réelles; il est clair que tout élément $\alpha\epsilon\Gamma$ définit d'une façon naturelle un homéomorphisme de X sur lui-même, qu'on notera aussi α; alors pour tout $f\epsilon A$ on a identiquement $f(\alpha\cdot x)=\alpha\cdot f(x)$; (4) enfin, quand f parcourt A la fonction $f(x)$ décrit l'ensemble de toutes les fonctions définies sur X, à valeurs dans Q, continues, nulles sur l'homomorphisme 0, et vérifiant la condition précédente. On obtient ainsi une caractérisation concrète des algèbres normées en question, résultat qui généralise des théorèmes bien connus de Stone et de Gelfand et Neumark (ceux-ci étant relatifs à des *-algèbres sur le corps complexe; l'auteur démontre du reste que toute

algèbre vérifiant ses conditions, et dans laquelle la multiplication par des scalaires complexes est possible, est commutative). L'auteur prouve aussi des résultats relatifs aux idéaux fermés dans une algèbre A vérifiant les conditions énoncées plus haut; un tel idéal est bilatère, c'est une intersection d'idéaux maximaux, et c'est aussi l'ensemble des f tels que la fonction $f(x)$ soit nulle sur un sous-ensemble fermé de X, invariant par les automorphismes de Q.

R. Godement (Nancy).

Kimura, Naoki. A note on normed ring. Kōdai Math. Sem. Rep., no. 3, 23–24 (1949).

The author gives a simple example of a normed ring with a left unit but no right unit. He also repeats the proof of Gelfand's theorem [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 3–24 (1941); these Rev. **3**, 51] that if R is a Banach space and a ring with a unit in which multiplication is continuous, then R is isomorphic to a normed ring where $\|xy\| \leqq \|x\|\|y\|$, in order to point out that Gelfand employed the existence of a right unit rather than the existence of a unit.

B. Yood (Ithaca, N. Y.).

Halilov, Z. I. Linear singular equations in a unitary ring. Mat. Sbornik N.S. **25**(67), 169–188 (1949). (Russian)

The author defines a unitary ring R to be a commutative algebra over the complex numbers in which there is defined an inner product (x, y) with the usual algebraic properties plus the condition that to each $y \epsilon R$ there exists at least one $\bar{y} \epsilon R$ such that $(x, yz) = (x\bar{y}, z)$ for all $x, z \epsilon R$. A unitary ring which is a Banach algebra under the norm $\|x\| = (x, x)^{\frac{1}{2}}$ is a special case of the H^*-algebras studied by Ambrose [Trans. Amer. Math. Soc. **57**, 364–386 (1945); these Rev. **7**, 126]. The present paper contains a development of the author's theory of ~~single~~ singular equations [cf., e.g., Doklady Akad. Nauk SSSR (N.S.) **60**, 1133–1136 (1948); these Rev. **10**, 48] for the case of unitary rings with an identity element.

C. E. Rickart (New Haven, Conn.).

Fort, M. K., Jr. A note on equicontinuity. Bull. Amer. Math. Soc. **55**, 1098–1100 (1949).

The author proves the following theorem. Let X and Y be compact metric spaces, let F be a set of continuous functions on X to X such that $x_1, x_2 \epsilon X$ implies $f(x_1) = x_2$ for some $f \epsilon F$, let G be a set of continuous functions on X to Y such that $f \epsilon F$ implies $g = hgf$ for some continuous function h on Y to Y and all $g \epsilon G$. Then G is equicontinuous. Corollary: the center of an algebraically transitive group of homeomorphisms on a compact metric space is equicontinuous. This answers affirmatively a question raised by the reviewer.

W. H. Gottschalk (Philadelphia, Pa.).

Friedlander, F. G. On the iteration of a continuous mapping of a compact space into itself. Proc. Cambridge Philos. Soc. **46**, 46–56 (1950).

The author considers topological questions of recursion and stability, suggested by the qualitative aspect of differential equations, in the context of a continuous transformation of a compact metric space into itself.

W. H. Gottschalk.

Rohlin, V. A. On the decomposition of a dynamical system into transitive components. Mat. Sbornik N.S. **25**(67), 235–249 (1949). (Russian)

The author's principal result is theorem 4 below; it was announced without proof in Doklady Akad. Nauk SSSR (N.S.) **58**, 189–191 (1947) [these Rev. **9**, 230]. He begins by defining a concept of measurability for many-valued functions from a measure space X into a metric space M. The important special case which motivates the discussion is the one in which M is the space $M(Y)$ of all measure-preserving transformations of a measure space Y into itself. (All the measure spaces discussed in the paper are what the author calls Lebesgue spaces, and are therefore measure-theoretically isomorphic to an interval together with a countable set of atoms.) Two functions f and g, from X into $M(Y)$, are [measurably] isomorphic if there exists a [measurable] function h such that $h(x)f(x)(h(x))^{-1} = g(x)$ for every x.

Theorem 1. If f is a single-valued measurable function from X into $M(Y)$, and if, for each x in X, $F(x)$ is the many-valued function which takes x into all measure-preserving transformations isomorphic to $f(x)$ (i.e., into the conjugate class of $f(x)$), then F is measurable. Theorem 2. If F is a many-valued measurable function from X into $M(Y)$, such that the set of values of $F(x)$ is, for each x in X, a conjugate class, then there exists a single-valued measurable function f from X into $M(Y)$ such that $f(x) \epsilon F(x)$ for every x in X. Theorem 3. If f and g are isomorphic measurable functions, then they are measurably isomorphic. Theorem 4. If for a fixed nowhere periodic measure-preserving transformation T (on a certain Lebesgue space), the space X is the base space of the canonical decomposition of T into indecomposable parts T_x, and if the (many-valued) function F_T maps each x into the conjugate class of T_x, then F_T is measurable and two measure-preserving transformations S and T are isomorphic if and only if the corresponding functions F_S and F_T are identical. (Since all the fibers Y_x are nonatomic and therefore isomorphic to each other, the values of F_T may be considered as if they were in a fixed Lebesgue space Y.) If, conversely, F is a many-valued measurable function from X into $M(Y)$, such that the set of values of F is the conjugate class of an indecomposable transformation, then there exists a measure-preserving transformation T such that $F_T = F$.

P. R. Halmos.

Halmos, Paul R. Measurable transformations. Bull. Amer. Math. Soc. **55**, 1015–1034 (1949).

A review of the progress made since 1931 in the study of measurable and measure-preserving transformations of measure spaces. The following topics are discussed: recurrence and individual ergodic theorems; indecomposable transformations, weak and strong mixing, and their generality; decomposition of a transformation into indecomposable components; relations between a transformation, the corresponding Boolean automorphism, and the induced unitary operator in L_2; conditions for the existence of an invariant measure; flows built under a function. A hitherto unpublished example, due to the author and J. von Neumann, of two nonisomorphic transformations which have spectrally equivalent operators is described. Several unsolved problems concerning isomorphism, the existence of invariant measures, and analogues of probability theorems are discussed. The paper concludes with a bibliography consisting of 136 titles.

J. C. Oxtoby (Bryn Mawr, Pa.).

Poritsky, Hillel. The billiard ball problem on a table with a convex boundary—an illustrative dynamical problem. Ann. of Math. (2) **51**, 446–470 (1950).

The relationship of Birkhoff's famous billiard ball problem to the transformation of a ring-shaped region R into itself is explained. In this connection several different sets of variables are used. Various forms of the integral invariant

of this transformation are also established. "Integrability" of the problem is defined as occurring when there exists a nonconstant function $\lambda(x, y)$ defined over R, such that $\lambda(x_1, y_1)=\lambda(x, y)$, where (x_1, y_1) is the point into which (x, y) is carried by the transformation. This concept of integrability is studied by considering the envelopes of families of chords drawn to the curve which bounds the "billiard table." It is shown, under certain conditions, that this bounding curve must be an ellipse if the problem is to be integrable. Questions of transitivity and finite intransitivity are also considered. The author states that "in the general nonintegrable case the transformation is transitive over the ring." But this has not, to the reviewer's knowledge, ever been proved, and if the statement is merely to be taken as the definition of the general nonintegrable case, it would be of the greatest interest to show that the "general" case may actually occur. Proof of the existence of transitivity in a case when the billiard table is symmetric about two perpendicular lines would probably also furnish the answer to the famous question as to whether formal linear stability in conservative holonomic dynamical systems implies actual stability. The answer would be in the negative.

D. C. Lewis (Baltimore, Md.).

Theory of Probability

***Kolmogorov, A. N. Foundations of the Theory of Probability.** Chelsea Publishing Company, New York, N. Y., 1950. viii+71 pp. $2.50.

Translation of Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933.

Rényi, Alfréd. 30 years of mathematics in the Sowiet Union. I. On the foundations of probability theory. Mat. Lapok 1, 27–64 (1949). (Hungarian. Russian and English summaries)

Gourevitch, Georges. Construction d'une loi de probabilité à partir d'une famille d'ensembles donnée. C. R. Acad. Sci. Paris 230, 170–172 (1950).

Let T_k be k-dimensional Euclidean space. For each a, $0\leqq a\leqq 1$, let P_a be a point set in T_k, monotone nondecreasing in a. The author considers the problem of defining a probability measure in T_k which assigns probability a to the set P_a, for all a. This is solved when $k=1$ if the P_a's are intervals or finite interval sums, and for $k>1$ in simple cases which can be reduced to the one-dimensional case by induction.

J. L. Doob (Ithaca, N. Y.).

***van Dantzig, D. Sur la méthode des fonctions génératrices.** Le Calcul des Probabilités et ses Applications. Colloques International du Centre National de la Recherche Scientifique, no. 13, pp. 29–45. Centre National de la Recherche Scientifique, Paris, 1949.

Using a personal terminology, the author studies sequences of events with the help of the linear form $\sum_n p_n A_n$ which, for $A_n = A^n$, becomes the usual generating function. Applications to the iteration problem, etc.

M. Loève.

Bottema, O. A probability computation of Emanuel Lasker. Simon Stevin 27, 1–5 (1949). (Dutch)

Two players alternate in tossing an m-faced die. After each toss a pawn, which can occupy one of the positions $0, 1, 2, \cdots$, is moved from its present position n to $|n-k|$, where k is the number scored. The player who first moves the pawn to 0 wins. The author finds a recurrence relation for the probabilities of winning and shows how to calculate them in terms of the roots of the characteristic equation.

W. Feller (Ithaca, N. Y.).

Tukey, John W. Moments of random group size distributions. Ann. Math. Statistics 20, 523–539 (1949).

Étant données N urnes $U_1, \cdots, U_i, \cdots$ et m boules, chacune de ces boules, par un tirage au sort et indépendamment des autres, est affectée à une urne dans laquelle on la place, ou encore n'est affectée à aucune urne; soit p_i la probabilité qu'une boule soit affectée à l'urne U_i $(\sum p_i \leqq 1)$ et $q=1-\sum p_i$ la probabilité qu'elle ne soit affectée à aucune urne; soit N_k le nombre des urnes qui se trouvent, l'opération terminée, contenir exactement K boules $(k=0, 1, \cdots, m)$. L'auteur, par l'emploi de "fonctions génératrices," calcule les moments (y compris les moments liés) de N_k; il examine le cas où $p_i=p$ est indépendant de i (et en particulier où $p\to 0$, $m\to+\infty$), puis le cas général (en particulier quand $p_i\to 0$ et $m\to+\infty$), résume ses résultats dans des tableaux et envisage des applications à une question de chimie et à une question de mutation de bactéries.

R. Fortet (Caen).

***Kac, Mark. Distribution problems in the theory of random noise.** Proc. Symposia Appl. Math. 2, 87–88 (1950). $3.00.

La théorie du "bruit de fond" présente des processus stochastiques $Z(t)$ du type: $Z(t)=\int_{-\infty}^{t} K(t-\tau)V[X(\tau)]d\tau$, où $X(t)$ est un processus laplacien stationnaire (bruit à la sortie d'un amplificateur linéaire), $V(x)$ la caractéristique d'un appareil non-linéaire (par exemple un détecteur quadratique, pour lequel $V(x)=x^2$), et où $K(u)$ (défini pour $u\geqq 0$) est la réponse percussionnelle d'un filtre linéaire. Envisageant le problème de déterminer la loi de probabilité de la variable aléatoire $Z(t)$, l'auteur rappelle les résultats qu'il a déjà obtenus [cf. pour le cas $V(x)=x^2$, Kac et Siegert, J. Appl. Phys. 18, 383–397 (1947); ces Rev. 8, 522; pour $V(x)$ quelconque, $K(u)=1$ pour $u<t$, $K(u)=0$ pour $u>t$, et $X(t)$ à accroissements indépendants, Kac, Trans. Amer. Math. Soc. 65, 1–13 (1949); ces Rev. 10, 383] et signale la possibilité, dans certains cas, de ramener le problème à la théorie générale des processus de Markoff et aux équations de Kolmogoroff.

R. Fortet (Caen).

Milicer Grużewska, Halina. The coefficient of correlation a posteriori of equivalent variables. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 39 (1946), 3–17 (1947). (English. Polish summary)

A. Khintchine [Rec. Math. [Mat. Sbornik] N.S. 12(54), 185–195 (1943); these Rev. 5, 168] considered sequences $x_1, x_2, \cdots$ of independent random variables and functions $f_1(x_1), f_2(x_2), \cdots$ of these variates. He determined the correlation coefficient for any pair $f_i(x_i), f_k(x_k)$ on condition that the value of the sum $x_1+\cdots+x_n$ is known. The author gives a modification of Khintchine's result. It is assumed that (*) $x_1+\cdots+x_n=a$, where a is a known constant but not a random variable and that $f_i(x_i)=f(x_i)$. Moreover the variates are not assumed to be independent. It is shown that under certain conditions $R_f{}^a-R_f=O(n^{-1})$. Here $R_f=R[f(x_i), f(x_k)]$, $i\neq k$, is the coefficient of correlation between $f(x_i)$ and $f(x_k)$ and $R_f{}^a$ is the coefficient of correlation between the same functions under the assumptions that (*) holds.

E. Lukacs (Washington, D. C.).

*Ottaviani, G. **La loi uniforme des grands nombres dans l'esprit de la théorie classique des probabilités. Considérations relatives au concept de nombre normal et aux liens avec la théorie de M. de Misès.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 11–17. Centre National de la Recherche Scientifique, Paris, 1949.

Expository paper which contains the usual remarks on the various topics announced in the title. *K. L. Chung.*

Erdös, P. Remark on my paper "On a theorem of Hsu and Robbins." Ann. Math. Statistics **21**, 138 (1950).

The paper appeared in the same Ann. **20**, 286–291 (1949); cf. these Rev. **11**, 40.

Erdös, P. On the strong law of large numbers. Trans. Amer. Math. Soc. **67**, 51–56 (1949).

Soit $f(x)$ une fonction réelle de la variable réelle x $(-\infty<x<+\infty)$, périodique de période 1 et telle que: $\int_0^1 f(x)dx=0$, $\int_0^1 f^2(x)dx=1$; $n_1<n_2<\cdots<n_k<\cdots$ est une suite indéfinie arbitraire de nombres positifs croissants tels que n_{k+1}/n_k reste supérieur à un nombre fixe $c>1$; soit $\Phi_n(x)$ la $n$ième somme partielle de la série de Fourier de $f(x)$; on sait que s'il existe un $\epsilon>0$ tel que $\int_0^1[f(x)-\Phi_n(x)]^2dx=O[(\log n)^{-\epsilon}]$, on a:

$$(1)\qquad \lim_{N\to+\infty} N^{-1}\sum_{k=1}^{N} f(n_k x)=0$$

pour presque tout x [cf. Kac, Salem et Zygmund, mêmes Trans. **63**, 235–243 (1948); ces Rev. **9**, 426], et aussi que si $n_k=2^k$, (1) a lieu pour toute $f(x)$. L'auteur donne un exemple d'une fonction $f(x)$ et d'une suite $\{n_k\}$ telles que $\lim_{N\to+\infty} N^{-1}\sum_{k=1}^{N} f(n_k x)=\infty$ pour presque tout x, et montre que (1) est valable s'il existe un $\epsilon>0$ tel que

$$\int_0^1[f(x)-\Phi_n(x)]^2dx=O[(\log\log n)^{-2-\epsilon}].$$

R. Fortet (Caen).

Prohorov, Yu. V. On the strong law of large numbers. Doklady Akad. Nauk SSSR (N.S.) **69**, 607–610 (1949). (Russian)

Let $\{\xi_n\}$ be a sequence of independent random variables, $\zeta_n=\xi_1+\cdots+\xi_n$, $\eta_n=n^{-1}\zeta_n$, $\chi_r=2^{-r}(\zeta_{2^{r+1}}-\zeta_{2^r})$. Denote the median of ξ by $m\xi$, the variance by $D\xi$. The author sketches a simple proof that $P(\eta_n-m\eta_n\to 0)=1$ if and only if (*) $\sum_{r=0}^{\infty}P(|\chi_r-m\chi_r|>\epsilon)<\infty$. It may be pointed out that analogous necessary and sufficient conditions for $P(\eta_n\to 0)=1$ can be inferred from a result by Kawata [Proc. Imp. Acad. Tokyo **16**, 109–112 (1940); these Rev. **1**, 340] who however failed to notice the (obvious) necessity, viz.: (a) $P(|\eta_n|>\epsilon)\to 0$ and (b) $\sum_{n=0}^{\infty}P(|\eta_{2^n}|>\epsilon)<\infty$; where (b) can be replaced by (b′) $\sum_{r=0}^{\infty}P\{|\chi_r|>\epsilon\}<\infty$ to look more like (*). Several consequences are stated, among which are: if either (i) each ξ_n has a Gaussian distribution or (ii) $\sup|\xi_n|=o(n(\log\log n)^{-1})$, then (*) can be replaced by $\sum_r e^{-\epsilon/D\chi_r}<\infty$. The proof of (ii) uses deep estimates by Kolmogorov [Math. Ann. **101**, 126–135 (1929)].

K. L. Chung (Ithaca, N. Y.).

Loève, Michel. Remarques sur la convergence presque sûre. C. R. Acad. Sci. Paris **230**, 52–53 (1950).

The following are among the results stated. Let $x_1, x_2, \cdots$ be any sequence of random variables. (1) If

$$\sum_1^{\infty}\int_0^{\epsilon} t\Pr\{|x_n|>t\}dt<\infty$$

then $\sum_1^{\infty}[x_n-\int_{-\epsilon}^{\epsilon}x d\Pr\{x_n\leqq x|x_1,\cdots,x_{n-1}\}]$ converges with probability 1. (2) If $\Pr\{c(x_n-x_k)\geqq 0|x_1,\cdots,x_k\}\geqq p>0$ for $c=\pm 1$, then $\Pr\{\max_{j\leqq n}|x_j|>\epsilon\}\leqq E\{|x_n|^r\}/p\epsilon^r$. The first result generalizes Kolmogorov's three-series theorem to dependent random variables; the second makes possible a generalization of Kolmogorov's sufficient condition for the strong law of large numbers in terms of second moments. (3) If $E\{|x_n|\}<\infty$, let A_m be an $x_1,\cdots,x_m$ set, and let $A_m^{(k)}$ be defined by the same conditions as A_m, but imposed on $x_k,\cdots,x_{k+m-1}$. Then if $A_m\downarrow 0$ implies

$$\lim_{m\to\infty}\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\Pr\{A_m^{(k)}\}=0$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|n^{-1}\sum_{k=1}^{n}\int_{A_m^{(k)}}x_k d\Pr\right|=0,$$

and if

$$\lim_{n\to\infty} n^{-1}E\{|x_n|\}=0,$$

then $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}x_k$ exists with probability 1.

J. L. Doob (Ithaca, N. Y.).

Juncosa, M. L. The asymptotic behavior of the minimum in a sequence of random variables. Duke Math. J. **16**, 609–618 (1949).

Soit $\{X_n\}$ $(n=1, 2, \cdots, \infty)$ une suite indéfinie de variables aléatoires indépendantes de fonctions de répartition $\{F_n(x)\}$, et posons: $Y_n=\min_{k\leqq n}X_k$. L'auteur étudie la fonction de répartition de Y_n pour $n\to+\infty$. (a) Il donne des conditions suffisantes pour qu'il existe une suite de constantes $\{a_n\}$ telles que $(Y_n-a_n)\to 0$ en probabilité, ou bien pour que, les a_n étant supposés positifs, $(Y_n-a_n)/a_n\to 0$ en probabilité. (b) Il donne des conditions suffisantes pour qu'il existe deux suites de constantes $\{a_n\}$ et $\{b_n\}$ telles que (1) $\lim_{n\to+\infty}\Pr(Y_n>a_nx+b)=e^{-P(x)}$, où $P(x)$ a une forme déterminée; si par exemple $P(x)$ est absolument continue, non-décroissante, positive pour $x>0$ avec $P(+0)=0$ et si $xP'(x)$ est non-décroissante pour $x\geqq 0$ avec $\lim_{x\to+\infty}xP'(x)=1$, il existe une suite $\{F_n(x)\}$ et une suite $\{a_n\}$ telles que (1) ait lieu avec $b_n=0$. (c) Dans le cas particulier où les $F_n(x)$ appartiennent au même type, il donne des conditions suffisantes pour que (1) ait lieu avec $P(x)=x^\tau$ $(\tau>0)$ ou $P(x)=\int_0^1 e^{xy}d\mu(y)$, où $\mu(y)$ est une fonction de répartition sur (0, 1). (d) Dans le cas général où les $F_n(x)$ ne sont pas nécessairement du même type, il donne des conditions suffisantes pour que (1) ait lieu avec $P(x)=x$. *R. Fortet.*

David, Herbert T. A note on random walk. Ann. Math. Statistics **20**, 603–608 (1949).

Let $G(x, y)$ be the expected number of steps in a discrete random walk to get outside an interval $(D-y, D)$. The author proves that, under specified conditions,

$$G(0,y)[G_{11}(x,y)+G_{02}(x,y)]=\frac{d}{dy}G(0,y)[G_{10}(x,y)+2G_{01}(x,y)].$$

M. Loève (Berkeley, Calif.).

Epstein, Benjamin. The distribution of extreme values in samples whose members are subject to a Markoff chain condition. Ann. Math. Statistics **20**, 590–594 (1949).

Observations x_k at time k, $k=1, 2, \cdots$ are taken from a stationary Markov process defined by $F_2(x, y)=P(x_i\leqq x, x_{i+1}\leqq y)$. Distributions of first and second smallest (also largest) values in samples of size n are obtained. [The author points out to the reviewer that his fundamental

The conditions (a) and (b′), but not (a) + (b) as given by Kawata, are necessary and sufficient for $P(\eta_n\to 0)=1$

assumption is not a Markov chain condition (as stated in the title and the text); condition (5) of the paper is what is actually assumed.] *M. Loève* (Berkeley, Calif.).

Franckx, Édouard. Relation entre les ensembles renouvelés et les probabilités en chaine. C. R. Acad. Sci. Paris **230**, 359–361 (1950).

A recurring series is defined by

$$u_n = qu_{n-1} + q^2 u_{n-2} + \cdots + q^r u_{n-r}. \tag{1}$$

An infinite sequence of vectors $\{V_n\}$ in r-dimensional space is associated with the recurrence formula (1). The ith coordinates of these vectors are the particular solutions of (1) obtained by using $u_j = \delta_{ij}$ $(i, j = 1, \cdots, r)$ as the initial vectors. Here δ_{ij} is the Kronecker symbol. The matrix of the vectors $[V_2, V_3, \cdots, V_{r+1}]$ defines a Markov chain. This Markov chain is used to study the asymptotic behaviour of the vectors of the sequence $\{V_n\}$.

E. Lukacs (Washington, D. C.).

Grenander, Ulf. Stochastic processes and integral equations. Ark. Mat. **1**, 67–70 (1949).

(1) For each $t\epsilon(a, b)$ let $x(t)$ be a random variable with $E\{x(t)\} \equiv m$ and $E\{[x(s)-m][x(t)-m]\} = r(s, t)$. It is supposed that $r(s, t)$ is continuous and known. The constant m is unknown and is to be estimated by random variables of the form $m^* = \int_a^b x(t)f(t)dt$, where f is quadratically integrable. A sequence of such estimates $\{m_n^*\}$ is found explicitly for which $\lim_{n\to\infty} E\{|m-m_n^*|^2\}$ is the greatest lower bound of $E\{|m-m^*|^2\}$ for admissible estimates m^*. (2) Problem (1) is generalized as follows. For each θ a probability measure is supposed given, and regularity conditions are imposed on the variation of this measure with θ. The value of θ is to be estimated from a sample. A necessary and sufficient condition is stated that there exist an unbiased estimate of θ (that is, one with the correct expectation for all θ). The condition is expressed in terms of the characteristic values and functions of a certain integral operator. (3) Let σ be a bounded measure of linear Borel t sets and let an $x(t)$ process be given as in (1). The author studies the integral equation $z = \lambda \int_{-\infty}^{\infty} x(t)f(t)d\sigma(t)$, finding expressions for the $x(t)$ process in terms of the solutions.

J. L. Doob (Ithaca, N. Y.).

***Fréchet, Maurice. Les valeurs typiques d'ordre nul ou infini d'un nombre aléatoire et leur généralisation.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 47–51. Centre National de la Recherche Scientifique, Paris, 1949.

For a more complete treatment see the author's paper in Ann. Sci. École Norm. Sup. (3) **65**, 211–237 (1948); these Rev. **10**, 386. *J. L. Doob* (Ithaca, N. Y.).

Mourier, Édith. Sur l'espérance mathématique d'un élément aléatoire dans un espace de Banach. C. R. Acad. Sci. Paris **229**, 1300–1301 (1949).

Fréchet [Revue Sci. (Rev. Rose Illus.) **82**, 483–512 (1944); these Rev. **8**, 141] defined the expectation $E\{X\}$ of a random variable X taking on values in a Banach space B as an abstract integral, the Bochner integral. The author uses the more general Pettis integral. The following form of the law of numbers is stated. Let $X_1, X_2 \cdots$ be mutually independent random variables taking on values in the separable Banach space B. It is supposed that the variables have a common distribution, with expectation the null element θ, and that $E\{\|X_n\|\} < \infty$. Then $(X_1 + \cdots + X_n)/n$ converges weakly to θ, with probability 1.

J. L. Doob (Ithaca, N. Y.).

Lévy, Paul. Fonctions aléatoires laplaciennes. C. R. Acad. Sci. Paris **229**, 1057–1058 (1949).

The author considers the equation

$$dZ(t) = dt \int_{t_0}^{t} f(t, u)dZ(u) + \sigma(t)dW(t)$$

for Z, where $W(t)$ is a Brownian motion and f and σ are given functions. The solution is a Gaussian process with $dW(t)$ independent of $Z(s)$ for $s<t$, and an integral equation for the covariance function is given. Necessary and sufficient conditions that the $Z(t)$ process be stationary are found. If $t_0 = 0$ and if $f(t, u)$ is defined for $0<t<u<2\pi$, there is a $P(t)$ process, $0<t<2\pi$, almost all of whose sample functions are continuous with period 2π, such that $Z(t)$ differs from $P(t)$ by a linear function. Then aside from this linear function $Z(t)$ has a Fourier expansion in $(0, 2\pi)$ with Gaussian coefficients. The Paley-Wiener expansion of a Brownian motion process is a special case. Another special case, depending on the two-parameter Brownian motion, is evaluated explicitly. *J. L. Doob* (Ithaca, N. Y.).

***Lévy, Paul. L'analyse harmonique des fonctions aléatoires stationnaires.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 111–120. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Expository paper. *J. L. Doob* (Ithaca, N. Y.).

***Wiener, Norbert. Sur la théorie de la prévision statistique et du filtrage des ondes.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 67–74. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Expository paper; see also the author's book [Extrapolation, Interpolation, and Smoothing of Stationary Time Series, Wiley, New York, 1949; these Rev. **11**, 118].

J. L. Doob (Ithaca, N. Y.)

Kosten, L., Manning, J. R., and Garwood, F. On the accuracy of measurements of probabilities of loss in telephone systems. J. Roy. Statist. Soc. Ser. B. **11**, 54–67 (1949).

The first few moments of the distributions of the number of lost calls and of the total time for which a full-availability group is blocked by congestion are determined with the aid of Laplace transforms. The distributions are characterized by the "coefficient of cluster effect," $f = (s^{-1} \operatorname{Var} s)^{\frac{1}{2}}$, in which s is the number of lost calls. For relatively small probability of loss, $f \approx (1+\eta)^{\frac{1}{2}}/(1-\eta)^{\frac{1}{2}}$, where η is the offered traffic per line. By comparison it is clear that counting lost calls does not give more information about the probability of loss than knowledge of the total time the system is blocked. The latter gives the greatest accuracy.

A. Jensen (Copenhagen).

TOPOLOGY

Bilinski, S., and Blanuša, D. Proof of the indecomposability of a certain graph. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 78–80 (1949). (Croatian)

The authors consider a certain graph. We may describe its structure by saying that its nodes can be numbered from 1 to 18 so that they are joined in pairs according to the following scheme: (1, 3), (1, 4), (1, 18), (2, 4), (2, 5), (2, 17), (3, 5), (3, 7), (4, 6), (5, 8), (6, 7), (6, 8), (7, 9), (8, 10), (9, 10), (9, 12), (10, 11), (11, 13), (11, 14), (12, 13), (12, 16), (13, 15), (14, 16), (14, 17), (15, 17), (15, 18), (16, 18). Thus three branches meet at each node. It is shown that the branches of the graph cannot be coloured in 3 colors so that no two of the same colour meet at any node. Two proofs are given. A direct proof considers the incidence matrix whose elements a_{jk} indicate whether and how the nodes j and k are joined. The indirect proof reduces the graph to two simpler ones. *W. T. Tutte* (Toronto, Ont.).

Tutte, W. T. The factorization of locally finite graphs. Canadian J. Math. **2**, 44–49 (1950).

The author extends the results of his earlier paper [J. London Math. Soc. **22**, 107–111 (1947); these Rev. **9**, 297] from finite graphs to infinite graphs in which each node is of finite degree. For instance, his final theorem may be expressed as follows. Let G be a connected graph with σ branches (or "links") at each of its n nodes, the number n being either even or infinite. Suppose further that G remains connected after the removal of any $\sigma-2$ nodes (along with all the branches belonging to them). Then the nodes of G can be paired in such a way that each pair consists of the two ends of a branch. *H. S. M. Coxeter.*

Frucht, Robert. Graphs of degree three with a given abstract group. Canadian J. Math. **1**, 365–378 (1949).

The group of a graph G is the group of all permutations of edges and vertices of G which preserve incidence relations (and which do not replace an edge by a vertex). The author considers the problem of finding a graph G whose group is simply isomorphic with a given abstract group $\mathfrak{H}$. He has shown in a previous paper that this problem is soluble [Compositio Math. **6**, 239–250 (1938)]. He now shows that it is soluble also when we impose the extra condition that each vertex of G shall be incident with just three edges. Let h be the order of $\mathfrak{H}$, and suppose that $\mathfrak{H}$ is generated by some n of its elements. The author's main result is that when $h>3$ we may take G to have $2h(n+2)$ vertices. If we relax the extra condition on G we may reduce this number to $2hn$ for noncyclic groups and $3h$ for cyclic groups. Thus the author has greatly improved on his previous value, $h(n+1)(2n+1)$, for the number of vertices required. *W. T. Tutte* (Toronto, Ont.).

Zarankiewicz, Casimir. Sur le nombre des points de ramification dans des dendrites et dans des graphes. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **39** (1946), 18–24 (1947). (French. Polish summary)

The author divides the nodes of a finite tree which are not incident with just two edges into two classes, nodes of degree 1 and nodes of degree greater than 2. He shows that the sum of the degrees of the nodes of the second class is equal to $2(n-1)+e$, where e and n are the numbers of nodes of the first and second classes, respectively. He obtains a generalization of this formula valid for all finite graphs. *W. T. Tutte* (Toronto, Ont.).

Sentis, Philippe. Quelques résultats relatifs au coloriage des cartes. C. R. Acad. Sci. Paris **230**, 355–357 (1950).

The author gives a necessary and sufficient condition that a given map of a finite number of regions, covering the plane, may be coloured in four colours. It is supposed as usual that just three edges and just three regions meet at each vertex. The condition is that the edges can be arranged in three mutually exclusive classes, A, B and C, so that (i) the number of members of $A\cup B$ incident with any region is even, and (ii) each vertex is incident with just two members of A. [This recalls P. G. Tait's condition, that the edges can be coloured in three colours, α, β and γ, so that no two of the same colour meet at any vertex. The two conditions are readily proved equivalent. ($B\cup C$ corresponds to the set of edges coloured γ.)] *W. T. Tutte* (Toronto, Ont.).

Artin, Emil. The theory of braids. American Scientist **38**, 112–119 (1950).

Cf. the author's papers [Ann. of Math. (2) **48**, 101–126, 643–649 (1947); these Rev. **8**, 367; **9**, 6].

Shimrat, Moshe. Non-decomposition of plane by simple arc. Riveon Lematematika **3**, 44–46, 52 (1949). (Hebrew. English summary)

Wang, Hsien-Chung. A new characterisation of spheres of even dimension. Nederl. Akad. Wetensch., Proc. **52**, 838–847 = Indagationes Math. **11**, 286–295 (1949).

This paper contains the following theorems. (I) Let W be a simply connected manifold with Euler characteristic equal to one. If W admits transitively a connected compact transformation group, then W is a single point. (II) Let W be a simply connected manifold with Euler characteristic equal to two. If W admits, effectively and transitively, a compact connected group R, then W is a sphere of even dimension, and R is either the orthogonal group or the exceptional simple Lie group G_2. The latter case occurs only when W is six-dimensional. (III) Let W be a simply connected manifold with Euler characteristic equal to a prime number $p>2$. If it admits transitively a compact group of transformations, then W is either a complex projective space of $2(p-1)$ dimensions or a quaternion projective space of $4(p-1)$ dimensions or a 16-dimensional closed orientable manifold with Poincaré polynomial $1+t^8+t^{16}$. The paper has some contact with work by Borel [Bull. Amer. Math. Soc. **55**, 580–587 (1949); these Rev. **10**, 680] and with work by Montgomery and Samelson [Ann. of Math. (2) **44**, 454–470 (1943); these Rev. **5**, 60]. *D. Montgomery.*

Iwamoto, Hideyuki. On integral invariants and Betti numbers of symmetric Riemannian manifolds. I. J. Math. Soc. Japan **1**, 91–110 (1949).

The author determines the Betti numbers of the following symmetric Riemannian spaces: $R(n, k)$ and $A(n, k)$, the real and complex Grassmann-manifold (k-planes in n-space); $S(n, k)$, the space of k-planes in complex n-space, invariant under $x\to I\bar{x}$, where I is a nondegenerate skew-symmetric matrix (even n, k); $S(n)$, the space of null-systems, with orthogonal (skew-symmetric) matrix (even n); $C(n)$, the

space of m-planes M in n-space, with $M \wedge \hat{M}=0$, where $\hat{M}=I(\hat{M})$ (even n); $A^+(n)$ and $A^-(n)$, the symmetric (respectively, skew-symmetric), unimodular unitary $n\times n$ matrices. In the first part the complete list of harmonics on $A^+(n)$, $A^-(n)$, $A(n, k)$ is determined, using the theory of weights, Young diagrams and reduction of Kronecker products. In particular, it is determined which irreducible representations of the transformation group of the three spaces (the unimodular unitary group $US(n)$) appear.

The second part contains the determination of the Betti numbers. The adjoint group of a symmetric space is the linear group, induced by the isotropy group; it is known that the pth Betti number equals the number of times the trivial representation is contained in the pth Grassmann or exterior power of the adjoint group. We describe the procedure for $R(n, k)$; all other cases follow similar lines. The adjoint group is found to be $S\rightarrow T_hST_k{}^t$, where S is the generic element of the space of $h\times k$ matrices ($h=n-k$), and T_h, T_k run through the orthogonal groups $O(h)$, $O(k)$, respectively. For the corresponding representation $S\rightarrow ASB^t$ of $GL(h)\times GL(k)$ Ehresmann has constructed the complete reduction of the pth Grassmann power. And for each irreducible component, the author has determined, in part 1, how often the trivial representation occurs, if one cuts down to $O(h)\times O(k)$. The connection with harmonics here comes from Cartan's theorem: for a homogeneous space the number of harmonic sets corresponding to a given irreducible representation is equal to the number of trivial representations ("linear invariants") contained in the representation cut down to the isotropy group. The results are as follows: the pth Betti number is, for $R(n, k)$, the number of partitions $p=f_1+\cdots+f_k$ ($h\geqq f_1\geqq\cdots\geqq f_k\geqq 0$, $k\geqq g_1\geqq\cdots\geqq g_h\geqq 0$), such that all f_i-f_k and g_i-g_k are even, where the g_i form the conjugate diagram to the f_i. For $A(n, k)$ it is, for even p, the number of partitions of $\frac{1}{2}p$ into k integers between 0 and h; for odd p, 0. For $S(n, k)$ it is, for $4|p$, the number of partitions of $\frac{1}{4}p$ into $\frac{1}{2}k$ integers between 0 and $\frac{1}{2}h$; otherwise 0. The Poincaré polynomial is for

$$\begin{aligned}
&S(n)\colon (1+z^3)(1+z^4)\cdots(1+z^{2(r-1)}), \quad n=2r;\\
&C(n)\colon (1+z^2)(1+z^4)\cdots(1+z^{2n});\\
&A^+(n)\colon (1+z^5)(1+z^9)\cdots(1+z^{4r+1}), \quad n=2r+1;\\
&\qquad\quad (1+z^5)(1+z^9)\cdots(1+z^{4r-3})(1+z^{2r}), \quad n=2r;\\
&A^-(n)\colon (1+z^5)(1+z^9)\cdots(1+z^{4r-3}), \quad n=2r+1 \text{ or } 2r.
\end{aligned}$$

The cases $A(n, k)$, $S(n)$, $C(n)$ had been determined previously by Ehresmann [Ann. of Math. (2) **35**, 396–443 (1934)]. *H. Samelson* (Ann Arbor, Mich.).

Reeb, Georges. Quelques propriétés globales des trajectoires de la dynamique dues à l'existence de l'invariant intégral de M. Élie Cartan. C. R. Acad. Sci. Paris **229**, 969–971 (1949).

Let E_1 be a vector field in a $2n$-manifold V_{2n}. The author studies the consequences of the existence of an integral invariant $\omega=\pi+Hdt$ of the differential system $dx=E_1(x)dt$ (π a Pfaffian, H a function on V_{2n}); this is suggested by the situation in classical dynamics; π and H are subject to certain restrictions. For a submanifold W_{2n-1}, defined by H=constant, assumed compact, the trajectories cannot have a compact cross section. If the trajectories are closed and define a fibration of W_{2n-1}, with base space W_{2n-2}, then the first Betti number of W_{2n-1} is zero; in W_{2n-2} a certain differential form can be defined from π, which gives W_{2n-2} a pre-Hermitian structure, and has topological consequences; the period function for the closed trajectories is constant on the components of W_{2n-2}. *H. Samelson.*

Lu, Chien-Ke. Classification of 2-manifolds with singular points. Bull. Amer. Math. Soc. **55**, 1093–1098 (1949).

It is shown that any closed, connected 2-manifold with a finite number of isolated singular points may be obtained by identifying certain points of a (not necessarily connected) 2-manifold without singularities. Using this fact, the author obtains a classification theorem for 2-manifolds with singularities, and determines their fundamental group and 1-dimensional homology group. *W. S. Massey.*

Chern, Shiing-Shen, and Sun, Yi-Fone. The imbedding theorem for fibre bundles. Trans. Amer. Math. Soc. **67**, 286–303 (1949).

Consider fiber bundles with given fiber F and group G of homeomorphisms of F. A fiber bundle with base space A is called universal if the classes of equivalent fiber bundles with given base space B correspond to the homotopy classes of mappings of B into A. [Compare N. E. Steenrod, Ann. of Math. (2) **45**, 294–311 (1944); these Rev. **5**, 214; L. Pontrjagin, C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 322–325 (1945); these Rev. **7**, 138.] To find a universal fiber bundle, it is enough to find one for the corresponding principal fiber bundle (with F, G replaced by G, G). This is done (allowing absolute neighborhood retracts for base spaces) in case the group is one of the classical linear groups. A corresponding problem for product bundles is treated.
H. Whitney (Cambridge, Mass.).

Chern, Shiing-Shen, and Hu, Sze-Tsen. Parallelisability of principal fibre bundles. Trans. Amer. Math. Soc. **67**, 304–309 (1949).

A fiber bundle is a product bundle if and only if the corresponding principal fiber bundle is. Assuming the base space is a complex, if a principal fiber bundle is a product bundle over the $(n-1)$-dimensional part, then it is also over the n-dimensional part if and only if a certain obstruction element vanishes. This element is characterized in terms of a certain cohomology group in the base space.
H. Whitney (Cambridge, Mass.).

Kudo, Tatsuji. Classification of topological fibre bundles. Osaka Math. J. **1**, 156–165 (1949).

Let f be a function on a connected topological space B having as values subsets of a topological space Ω. If F is a topological space, f is defined to be F-continuous if (a) for each $b\epsilon B$, $f(b)$ is homeomorphic to F; (b) for each $b_0\epsilon B$ there are a neighborhood V of b_0 and a family $\{\sigma_b\}$, $b\epsilon V$, of homeomorphisms such that σ_b carries $f(b_0)$ onto $f(b)$ and σ_{b_0}=identity; and (c) the correspondence carrying $(y, b)\epsilon f(b_0)\times V$ into $(\sigma_b y, b)$ is a homeomorphism. Two F-continuous functions are F-homotopic in Ω if there is an F-continuous homotopy joining them. In a natural way, the graph of an F-continuous function is a topological fibre bundle over B with fibre F (i.e., a fibre bundle whose group is the group G of all homeomorphisms of F). Under rather weak restrictions on B, F and Ω the author proves several theorems, notably the following. (III) The topological fibre bundles over B are in one-to-one correspondence with the set of F-homotopy classes of F-continuous functions on B to the fundamental cube E_ω in Hilbert space. [This is an analogue of a theorem of Steenrod, Ann. of Math. (2) **45**, 294–311 (1944); these Rev. **5**, 214.] (V) If B' is obtained from $B\times$(unit interval) by identifying as points the sets $B\times\{0\}$ and $B\times\{1\}$ then the set of all topological fibre bundles over B' with fibre F is in one-to-one correspondence with the set of all equivalence classes of continuous func-

tions of B into the group G of homeomorphisms of F onto itself. [This generalizes a theorem of Feldbau, C. R. Acad. Sci. Paris **208**, 1621–1623 (1939).] (VI) The $(n+1)$th F-homotopy group of E_∞ (based on F-continuous functions on the $n+1$ sphere to E_∞) is isomorphic to $\pi_n(G)$.

J. L. Kelley (Berkeley, Calif.).

Hirsch, Guy. L'anneau de cohomologie d'un espace fibré et les classes caractéristiques. C. R. Acad. Sci. Paris **229**, 1297–1299 (1949).

Let $\bar{M}$ be a fibre bundle with base space M and fibre F. The author discusses a special case of the general problem of determining the cohomology ring of $\bar{M}$ in terms of the cohomology rings of F and M and certain additional invariants of the structure of the bundle. The methods used are based on a previous note of the author [same C. R. **227**, 1328–1330 (1948); these Rev. **10**, 558]. *W. S. Massey.*

Hirsch, Guy. Sur la structure multiplicative de l'anneau de cohomologie d'un espace fibré. C. R. Acad. Sci. Paris **230**, 46–48 (1950).

This is a continuation of a previous note by the same author [see the preceding review]; he discusses other special cases of the problem of determining the cohomology ring of a fibre bundle in terms of the cohomology rings of the base space and fibre. *W. S. Massey* (Princeton, N. J.).

Eilenberg, Samuel, and MacLane, Saunders. Homology of spaces with operators. II. Trans. Amer. Math. Soc. **65**, 49–99 (1949).

In einem zusammenhängenden Raume X, dessen Homotopiegruppen π_n für $1<n<q$ verschwinden, sind die Homologiegruppen H_n, $n<q$, durch π_1 bestimmt, und ebenso die Faktorgruppe H_q/Σ_q, wo Σ_q die aus den sphärischen Homologieklassen bestehende Untergruppe von H_q ist (das Bild von π_q vermöge des natürlichen Homomorphismus h von π_q in H_q). In der vorliegenden Arbeit wird einleitend durch einfache Beispiele belegt, dass H_q selbst nicht bestimmt ist, auch wenn π_q und die Operationen von π_1 in π_q bekannt sind; das Hauptziel der Arbeit ist die Definition einer neuen topologischen Invarianten k^{q+1} von X, welche zusammen mit π_1 und π_q für beliebige Koeffizientenbereiche G die Gruppe H_q (oder, was auf dasselbe hinausläuft, die Cohomologiegruppe H^q von X) bestimmt, und es wird eine explizite algebraische Konstruktion von H^q aus π_1, π_q, k^{q+1} und G angegeben. Es ist k^{q+1} ein Element der algebraischen Cohomologiegruppe $H^{q+1}(\pi_1, \pi_q)$, wie sie von den Verfassern und Eckmann eingeführt wurde. Der Fall $q=1$, wo also über die Homotopiegruppen von X nichts vorausgesetzt wird, ist von den Verfassern schon früher [Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946); diese Rev. **8**, 398] mit Homotopiemethoden hergeleitet worden. Die jetzige Darstellung für beliebiges $q\geqq 1$ geht von X zur universellen Überlagerung $\tilde{X}$ über; diese besitzt π_1 als fixpunktfreie Operatorgruppe, und es können die Homologiemethoden zur Anwendung kommen, welche von Eilenberg im Teile I der Arbeit [Trans. Amer. Math. Soc. **61**, 378–417 (1947); diese Rev. **9**, 52] für Räume und Komplexe mit Operatoren entwickelt wurden, insbesondere die "equivarianten" singulären Cohomologiegruppen $\tilde{H}_e{}^n$ von $\tilde{X}$ bezüglich eines Koeffizientenbereiches G mit π_1 als Operatorgruppe (sind die Operationen trivial, so ist $\tilde{H}_e{}^n$ isomorph zu H^n, andernfalls zur n-ten Cohomologiegruppe von X mit lokalen Koeffizienten im Sinne von Steenrod). Die gewöhnlichen Homologiegruppen $\tilde{H}_n$ von $\tilde{X}$ sind $=0$ für $n<q$, und $\tilde{H}_q$ ist operatorisomorph zu π_q. Die neue Invariante k^{q+1} tritt als "Hindernis" $\varepsilon\ H^{q+1}(\pi_1, \tilde{H}_q)$ bei der Erweiterung einer gewissen Homologie-Abbildung auf, und das Hauptergebnis (welches ganz allgemein für Komplexe mit Operatoren gilt, die in den Dimensionen $n<q$ azyklisch sind) liefert die explizite algebraische Bestimmung von $\tilde{H}_e{}^q$ aus π_1, $\tilde{H}_q$, k^{q+1} und G. Der zugehörige Formalismus wird auf verschiedene Arten begründet und diskutiert, speziell im Falle trivialer Operationen von π_1 in G, wo sich Eigenschaften des Homomorphismus h von π_q auf H_q ergeben. Die neue Invariante wird in ausführlicher Weise für die verallgemeinerten Linsenräume bestimmt und mit andern Ergebnissen über diese verknüpft. *B. Eckmann* (Zürich).

Giever, John B. On the equivalence of two singular homology theories. Ann. of Math. (2) **51**, 178–191 (1950).

Let G (G') be a topological (discrete) Abelian group, and $H^n(X, G; \mathfrak{U})$ $[H_n(X, G'; \mathfrak{U})]$ the cohomology (homology) groups of an arbitrary space X over G (G') based on cardinal number $\mathfrak{U}$ [Hurewicz, Dugundji, and Dowker, same Ann. (2) **49**, 391–406 (1948); these Rev. **9**, 606]. The homology groups are known to be all isomorphic to $H_n(X, G'; \aleph_0)$ while the cohomology groups depend on the cardinal number $\mathfrak{U}$, being all isomorphic if $\mathfrak{U}$ is sufficiently large. The author shows that, if $\mathfrak{A}$ is the cardinal number of the set of points of X, then the cohomology groups (and also the homology groups) based on any cardinal greater than $\mathfrak{A}^{\aleph_0}$ are in fact all isomorphic with the singular connectivity groups of Eilenberg [same Ann. (2) **45**, 407–447 (1944); these Rev. **6**, 96]. The method consists in constructing a polytope P which is very intimately related to the space X, having the same combinatorial and, in fact, also the same homotopy structure as that of the space X, as follows. Let $S(X)$ denote the Eilenberg singular complex of X. From each element of $S(X)$ choose a representative $T^n: s_n\rightarrow X$, s_n an ordered simplex; form the complex $R(X)$ composed of disjoint copies of the s_n selected, and define $T: R(X)\rightarrow X$ by $T|s_n=T^n$. The vertices of $R(X)$ can be given an ordering in which the vertices of each s_n are consecutive. Two simplexes (or faces of simplexes) of $R(X)$ are identified if they are mapped in the same way, that is, if there is an order-preserving isomorphic mapping B of one on the other such that $TB(p)=T(p)$, $p\varepsilon s$. A suitable subdivision of the set P so obtained yields the polytope in question. The assertion is proved by showing the combinatorial connectivity groups of P are the same as those of $S(X)$, and (longer and more difficult) the same as the Hurewicz-Dugundji-Dowker groups of X (with the mentioned restriction on the cardinal). It is further shown that the homotopy groups of P are those of X. Use of the polytope P enables the author to extend the Eilenberg-MacLane universal coefficient theorem [same Ann. (2) **43**, 757–831 (1942); these Rev. **4**, 88] to the Hurewicz-Dugundji-Dowker groups: $H_n(X, I; \aleph_0)$, I the integers, determine all the groups $H_n(X, G'; \aleph_0)$ and $H^n(X, G; \mathfrak{U})$ for all $\mathfrak{U}>\mathfrak{A}^{\aleph_0}$. *J. Dugundji* (Los Angeles, Calif.).

Whitehead, J. H. C. The secondary boundary operator. Proc. Nat. Acad. Sci. U. S. A. **36**, 55–60 (1950).

Let K be a polytope, $\pi_1(K)=0$, and K^n its n-skeleton. By introducing a group $\Gamma_n\subset\pi_n(K^n)$, the image of the natural homomorphism $\pi_n(K^{n-1})\rightarrow\pi_n(K^n)$, $n\geqq 2$, the author embeds the natural homomorphisms $j: \pi_n(K)\rightarrow H_n(K)$ into an exact sequence as follows:

$$s(k): \cdots\rightarrow H_{n+1}\xrightarrow{b}\Gamma_n\xrightarrow{i}\pi_n\xrightarrow{j}H_n\rightarrow\cdots\rightarrow H_3\rightarrow 0\rightarrow\pi_2\rightarrow H_2\rightarrow 0,$$

where i is the homomorphism $\pi_n(K^n)\to\pi_n(K)$ restricted to Γ_n, and b is the boundary homomorphism $\pi_{n+1}(K^{n+1}, K^n)\to\pi_n(K^n)$ restricted to the cycles in $C_{n+1}(K)=\pi_{n+1}(K^{n+1}, K^n)$. Let $S_m(K)$ be that portion of this exact sequence starting with H_m. The homomorphism of one sequence into another being defined as usual to consist of a family of homomorphisms $h:H_n\to H_n'$, $g:\Gamma_n\to\Gamma_n'$, $f:\pi_n\to\pi_n'$ which commute in each square, a $\varphi:K\to K'$ induces a homomorphism $S(K)\to S(K')$; from previous results of the author [Bull. Amer. Math. Soc. 55, 213–245 (1949); these Rev. 11, 48] if dim K, dim $K'\leqq m$ and $\varphi:K\to K'$ induces an isomorphism on, K and K' belong to the same homotopy type.

Using $\lambda:S^3\to S^2$ to represent a generator of $\pi_3(S^2)$ and $\mu:S^2\to K$ to represent $a\epsilon\pi_2(K)$, $\mu\lambda$ represents $\lambda(a)\epsilon\Gamma_3$; a homomorphism $S(K)\to S(K')$ is proper if $g\lambda(a)=\lambda f(a)$. If dim $K\leqq 4$ any proper (algebraic) homomorphism $S_4(K)\to S_4(K')$ is induced by a $\varphi:K\to K'$. In $S_4(K)$, π_3 is an extension of H_3 by Γ_3/bH_4 and $S_4(K)$ is determined (algebraically) up to a certain type of isomorphism, by H_2, H_3, H_4, the homomorphism $b:H_4\to\Gamma_3$ and the element of $H^2(H_3, \Gamma_3/bH_4)$ which determines the equivalence class of the extension π_3. The rest of the paper is devoted to giving details and properties of a group construction that will help calculate $S_4(K)$ algebraically, for K a finite simply connected polytope. For an A_n^2-polytope (i.e., a finite polytope K, dim $K\leqq n+2$, and $\pi_i(K)=0$, $i<n$) it follows that $S_{n+2}(K)$ is determined by its "A_n^2-ring." *J. Dugundji* (Los Angeles, Calif.).

Pontryagin, L. S. On a connection between homology and homotopy. Amer. Math. Soc. Translation no. **11**, 15 pp. (1950).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Math. **13**, 193–200 (1949); these Rev. **11**, 122.

Chen, Chieh. A note on the classification of mappings of a $(2n-2)$-dimensional complex into an n-sphere. Ann. of Math. (2) **51**, 238–240 (1950).

The author shows that some recent theorems of Hu [same Ann. (2) **50**, 158–173 (1949); these Rev. **10**, 393] on the homotopy classification problem for the case of mappings of a complex X of dimension less than $2n-2$ into an n-sphere S^n still remain true in case dimension $X=2n-2$.
W. S. Massey (Princeton, N. J.).

Morse, Marston. L-S-homotopy classes on the topological image of a projective plane. Bull. Amer. Math. Soc. **55**, 981–1003 (1949).

The author has previously [Ann. Soc. Polon. Math. **21** (1948), 236–256 (1949); these Rev. **11**, 123] determined models for the L-S- (locally simple) homotopy classes of closed p- (parameterized) curves on closed orientable surfaces. The present paper is concerned with the case where the surface S under consideration is the topological image of a projective plane. If f is a closed p-curve on S, $f\approx 0$ shall mean that f is homotopic to zero. The main result is the following theorem. Let h be a simple closed p-curve on S with $h\not\approx 0$. If n is a positive integer, let $h^{(n)}$ be a closed p-curve on S which traces h n times. Then any L-S-closed p-curve f on S is in the L-S-homotopy class of $h^{(1)}$ or $h^{(3)}$ if $f\not\approx 0$, and of $h^{(2)}$ or $h^{(4)}$ if $f\approx 0$. No two of the p-curves $h^{(1)}$, $h^{(2)}$, $h^{(3)}$, $h^{(4)}$ are in the same L-S-homotopy class.

By introduction of the sphere M which is the universal covering manifold of S and the natural mapping of M onto S the problem of determining the L-S-homotopy classes on S can be reduced to the study of L-S-homotopy classes (possibly restricted) on M, and use can be made of the previous results obtained for the sphere. In order to complete the proof and, in particular, in order to show that $h^{(1)}$ and $h^{(3)}$ are in different L-S-homotopy classes, a new concept of S-difference order $d_S(f)$, in case $f\not\approx 0$, is introduced and it is proved that f is L-S-homotopic to $h^{(1)}$ or $h^{(3)}$ according as $d_S(f)=1$ or 3 (mod 4). *G. A. Hedlund.*

Scorza Dragoni, Giuseppe. Su una questione di topologia. Giorn. Mat. Battaglini (4) **2**(78), 121–127 (1949).

Let K be a closed subset of $I=I_{n-r}\times I_r$ (I_k a unit k-cube). Let $s(x)=(x\times I_r)\cap K$, $x\epsilon I_{n-r}$. Assume that $s(x)$ satisfies the following continuity condition: if $\sigma(x)$ is a closed subset of $s(x)$ such that $s(x)-\sigma(x)$ is closed, then for y sufficiently near x there exist points of $s(y)$ within a preassigned distance of $\sigma(x)$. Theorems: if $r=1$, every arc in I joining $I_{n-1}\times 0$ with $I_{n-1}\times 1$ meets K; if $r=n-1$, every subcontinuum of I separating $0\times I_{n-1}$ from $1\times I_{n-1}$ meets K.
P. A. Smith (New York, N. Y.).

Trevisan, Giorgio. Su una questione di topologia. Rend. Sem. Mat. Univ. Padova **18**, 231–233 (1949).

A simple proof of the theorem of Scorza Dragoni with $r=1$ quoted in the preceding review. *P. A. Smith.*

Magenes, Enrico. Proprietà topologiche di certi insiemi di punti e teoremi di esistenza di punti uniti in trasformazioni plurivalenti di una r-cella in sè. Giorn. Mat. Battaglini (4) **2**(78), 168–181 (1949).

We use the notation of the two preceding reviews. Let F be the totality of points $x\times f(x)$ where f is a continuous function on I_{n-r} to I_r. Assume that the sets $s(x)$ satisfy the continuity condition of the theorem of Scorza Dragoni, and that $1<r<n-1$. In general F need not intersect K unless the sets $s(x)$ satisfy further conditions. With the aid of fixed-point theorems of Eilenberg and Montgomery [Amer. J. Math. **68**, 214–222 (1946); these Rev. **8**, 51] and of Hamilton [Duke Math. J. **14**, 689–693 (1947); these Rev. **9**, 197] it is shown that either one of the following additional conditions will ensure an intersection: (1) the sets $s(x)$ are acyclic relative to some coefficient field; (2) each $s(x)$ is the boundary of an r-cell which contains an r-sphere of diameter greater than d, where d is a positive number independent of x.
P. A. Smith (New York, N. Y.).

Magenes, Enrico. Un criterio di esistenza di punti uniti in trasformazioni topologiche piane. Rend. Sem. Mat. Univ. Padova **18**, 68–114 (1949).

Let Γ be a closed curve in the Euclidean plane π. It is assumed that Γ admits at most a finite number of multiple points, that its multiple points are of finite multiplicities, and that it is reducible to a simple closed curve by arbitrarily small modifications in the neighborhood of its multiple points. Let t be a topological mapping $\pi\to\pi$ and let $\Gamma'=t\Gamma$. Let J be the region of points of nonzero index relative to Γ. The main result is a theorem stating a set of conditions which is sufficient to ensure the existence of a fixed point for $t|(\Gamma+J(\Gamma))$. The conditions are (1) the existence of a point O common to $J(\Gamma)$ and $J(\Gamma')$; (2) the nonintersection of certain arcs of Γ (depending on O and on the intersection of Γ and Γ') with their respective images. When Γ is a simple closed curve, the theorem reduces to a theorem of Scorza Dragoni [Ann. Mat. Pura Appl. (4) **25**, 43–65 (1946); these Rev. **9**, 455]. *P. A. Smith* (New York, N. Y.).

Ghezzo, S. Intorno ad un teorema sulle quasi-traiettorie di una traslazione piana generalizzata. Rend. Sem. Mat. Univ. Padova **18**, 177–180 (1949).

Let t be a topological orientation-preserving fixed point-free automorphism of the plane π. Let $I\subset\pi$ be a closed bounded set which can be approximated (in a certain sense) to an arbitrary degree by translation arcs of t. Let $\beta\subset\pi$ be a simple arc whose ends are in $tI\cup t^2I\cup\cdots$ and $t^{-1}I\cup t^{-2}I\cup\cdots$ respectively and which, except possibly for its ends, fails to meet I. Conclusion: β intersects $t\beta$. The proof makes use of certain results of Scorza Dragoni [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 156–161 (1946); these Rev. **8**, 285]. *P. A. Smith.*

Satō, Tokui. On a fixed point theorem. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 33–44 (1949). (Esperanto)

The fixed-point theorems of Birkhoff-Kellogg, Schauder and Tychonoff have been used to establish the existence of solutions of various types of functional equations. The author utilizes the method of Tychonoff and an extension of the concept of a "locally convex" topological space to generalize the latter's theorem. From this result, several other fixed-point theorems are derived. *R. Bellman.*

Nagumo, Mitio. Sufficient conditions for a locally topological mapping to be univalent. J. Osaka Inst. Sci. Tech. Part I. **1**, 33–35 (1949). (Esperanto)

Two theorems on one-to-one mappings are proved, of which the following is typical. Let R be a metric space and let S be a connected metric space. Let D be a connected open subset of R such that D^- is compact. Let f be an open locally homeomorphic mapping of D^- into S. Suppose that (1) $f(D)$ is a proper subset of S; (2) the set $D^-\cap D'$ is connected; (3) f is a one-to-one mapping on $D^-\cap D'$. Then f is a one-to-one mapping throughout D^-. *E. Hewitt.*

Smirnov, Yu. On systems of coverings of topological spaces. Doklady Akad. Nauk SSSR (N.S.) **69**, 611–613 (1949). (Russian)

Let X be a topological space, and let $\mathfrak{A}$ and $\mathfrak{B}$ be finite open coverings of X. The covering $\mathfrak{A}$ is said to be inscribed in $\mathfrak{B}$ if, for every $A\in\mathfrak{A}$, there exists a $B\in\mathfrak{B}$ such that $A\subset B$. A collection Σ of finite open coverings of X is said to be defining if, for every finite open covering $\mathfrak{X}$ of X, there exists a covering $\mathfrak{S}\in\Sigma$ such that $\mathfrak{S}$ is inscribed in $\mathfrak{X}$. The least cardinal number of a defining collection Σ is called the combinatorial character of X and is denoted by the symbol $\sigma(X)$. The following theorems are proved. (I) An infinite regular space X is a compact metric space if and only if $\sigma(X)=\aleph_0$. (II) If X is a separable noncompact metric space, then $\sigma(X)=2^{\aleph_0}$. [Reviewer's note: this result is true for all normal spaces X with countable open bases.] (III) For every normal space X, $\sigma(X)=\sigma(\beta X)$ (βX being the Stone-Čech compactification of X). Other results concerning βX for normal spaces X are proved; most of these can be easily verified from the usual definition of βX. *E. Hewitt.*

Iseki, Kiyosi. On paracompact spaces. J. Osaka Inst. Sci. Tech. Part I. **1**, 67–74 (1949). (Japanese)

Expository note. Report on recent works on paracompact spaces by J. Dieudonné [J. Math. Pures Appl. (9) **23**, 65–76 (1944); these Rev. **7**, 134], C. H. Dowker [Duke Math. J. **14**, 639–645 (1947); these Rev. **9**, 196] and A. H. Stone [Bull. Amer. Math. Soc. **54**, 977–982 (1948); these Rev. **10**, 204]. *S. Kakutani* (New Haven, Conn.).

Keldyš, Lyudmila. Zero-dimensional mappings of finite-dimensional compacta. Doklady Akad. Nauk SSSR (N.S.) **68**, 989–992 (1949). (Russian)

Let X and Y be compacta of dimension n and $m>n$, respectively, and let $f(X)=Y$ be a continuous transformation whose inverse-sets, $f^{-1}(y)$, $y\in Y$, are all zero-dimensional. The author finds a property of the transformation f (and the spaces X and Y) which is sufficient that f shall be represented as the product of a finite number of simpler transformations. Those transformations are either of multiplicity two (inverse sets contain no more than two points) or they do not increase dimension. The property is called property (γ) and is as follows: to every open set V of X such that $\dim X<\dim f(V)=k$ and for every i for which $\dim X<i\leqq k$ there can be found in V a perfect set $Q=Q(i,V)$ such that $\dim f(Q)=i$ and $\dim f(Q\cdot d)\cdot f(Q\cdot d')\leqq i-1$ for every pair of nonintersecting pieces (closures of open sets) d and d' of X. This is a generalization of an earlier note, to which the author refers at several points for details of proof [same Doklady (N.S.) **66**, 327–330 (1949); these Rev. **11**, 45]. *L. Zippin* (Flushing, N. Y.).

Fort, M. K., Jr. A proof that the group of all homeomorphisms of the plane onto itself is locally arcwise connected. Proc. Amer. Math. Soc. **1**, 59–62 (1950).

Let P be the Euclidean plane, let H be the group of all homeomorphisms of P onto P, and let H be provided with its compact-open topology. The author proves by means of an isotopy construction that H is locally arcwise connected. *W. H. Gottschalk* (Philadelphia, Pa.).

Harrold, O. G., Jr. Euclidean domains with uniformly Abelian local fundamental groups. Trans. Amer. Math. Soc. **67**, 120–129 (1949).

An open subset A of the n-dimensional sphere S^n is said to be unalg (have uniformly Abelian local fundamental groups) if, given any neighborhood U of any point p of S^n there is a smaller neighborhood V of p such that every closed path in $A\cap V$ which represents a commutator in $\pi_1(A\cap U)$ is contractible in $A\cap U$. It is shown that an unalg open subset A of S^n is simply connected if its complement S^n-A is either a compact totally disconnected set or is a k-cell for some $k=1,2,\cdots,n$. If one of the components, A, of the complement of an $(n-1)$-dimensional sphere in S^n is unalg then both A and $\bar{A}$ are contractible. In connection with the unalg property, homology and homotopy uniform local connectedness in dimensions zero and one are considered. *R. H. Fox* (Princeton, N. J.).

Hopf, Eberhard. A theorem on the accessibility of boundary parts of an open point set. Proc. Amer. Math. Soc. **1**, 76–79 (1950).

It is shown that if G is a bounded open set in Euclidean n-space and R is a region in G whose boundary has a non-empty set C in common with the boundary of G, then C contains a set accessible from within G; and if C is the union of two disjoint closed sets, each of these sets likewise contains a set accessible from within C. A similar conclusion is obtained with assumptions on R stated in terms of local connection to the set C rather than having R itself connected.

It may be remarked that the notions, methods and results involved here are closely related to studies on accessibility made by several mathematicians twenty odd years ago; and it seems that parts at least of the author's conclusions, including the special cases he refers to as necessary to bridge a gap in the proof of a classical theorem on surfaces on non-

positive Gaussian curvature, are contained in these earlier studies. As an example, comparison might be made with a result of Kuratowski and Knaster [Fund. Math. 5, 23–58 (1924), in particular, p. 38] and with a paper of the reviewer [ibid. 14, 311–326 (1929)]. *G. T. Whyburn.*

Myškis, A. D. On the concept of boundary. Mat. Sbornik N.S. 25(67), 387–414 (1949). (Russian)

The author states that his object is to generalize, by means of ideas similar to those of Carathéodory's theory of prime ends, a number of existing non-equivalent conceptions of boundary, due to Carathéodory, Perkins, Kaufmann, Mazurkiewicz, Freudenthal, etc., which he discusses specially at the end of the paper. He lays down under what conditions a set of certain ideal elements (of type M or N or P) is to be regarded as constituting a boundary (not necessarily unique) of a given topological space R. Here M is used for a class whose members are non-empty open subsets G of R, P for one whose members are sequences $\{x_n\}$ of points of R, N for one whose members are, more generally, of the form $\{x_\alpha\}$ where α is an element of an ordered set Θ. Only the case of ideal elements of type M will be dealt with in this review; and in what follows $M(x)$ denotes the class of neighbourhoods G of any point $x\epsilon R$; a class M_1 is a refinement of a class M_2 if for each $G_2\epsilon M_2$ there is a $G_1\epsilon M_1$ such that $G_1\subset G_2$; a class M is completely regular if for each $G\epsilon M$ there is a real function $f(x)$, continuous in R, which vanishes outside G and equals 1 throughout some $G'\epsilon M$; it is termed "central" if the members of every finite subclass have a non-empty intersection; it is termed a "completely regular end" if it is central and completely regular without being a proper subclass of a central completely regular class.

A set $\mathfrak{M}$ of the classes M is termed "boundary in the wide sense" of R, if (1) for each $M\epsilon\mathfrak{M}$ the intersection of any two members G_1, G_2 of M contains as subset a member of G of M, and (2) for different classes M_1, M_2 in $\mathfrak{M}$, and also for each M_1 in $\mathfrak{M}$ and each M_2 of the form $M(x)$, there exist disjoint G_1, G_2 such that $G_1\epsilon M_1$, $G_2\epsilon M_2$. In the case in which R is completely regular, $\mathfrak{M}$ is termed "boundary in the narrow sense" provided that the following additional assumptions are satisfied: (3) each $M\epsilon\mathfrak{M}$ is completely regular; (4) each completely regular end is a refinement of some $M\epsilon\mathfrak{M}$; (5) for each $M=M(x)$ there is $G\epsilon M$ so that for every $M_1\epsilon\mathfrak{M}$ the relation $GG_1=0$ holds for some $G_1\epsilon M_1$; (6) for each $G'\epsilon M\epsilon\mathfrak{M}$ there is $G\epsilon M$ so that for every $M_1\epsilon\mathfrak{M}$ one at least of the relations $GG_1=0$, $G\subset G'$ holds for some $G_1\epsilon M_1$.

The topology of $R+\mathfrak{M}$ is defined by introducing as neighbourhoods (of each of their elements) in $R+\mathfrak{M}$, the sets H^* obtained by adding to each open $H\subset R$ the set of $M\epsilon\mathfrak{M}$ such that $G\subset H$ for some $G\epsilon M$. Compactification theorems are proved which justify these definitions. *L. C. Young.*

Koseki, Ken-iti. Über die Begrenzung eines besonderen Gebietes. II. Proc. Japan Acad. 21 (1945), 385–391 (1949).

Koseki, Ken-iti. Über die Begrenzung eines besonderen Gebietes. II. Jap. J. Math. 19, 345–369 (1948).

[For part I see Proc. Imp. Acad. Tokyo 20, 406–408 (1944) and Proc. Japan Acad. 21 (1945), 285–299 (1947); these Rev. 7, 335; 10, 389. The first paper under review is a summary of the second.] Suppose r is the common boundary of two plane domains G and G', and suppose G is bounded. Theorem 1. In order that r be monostratic it is necessary and sufficient that G have uncountably many simple prime ends. Theorem 2. If r is cyclically decomposed in "tranches," and if uncountably many of these contain points which are accessible from G, then r contains uncountably many points each of which is accessible from all sides from G. For related work and definitions see Kuratowski [Fund. Math. 12, 20–42 (1928)] and Carathéodory [Math. Ann. 73, 323–370 (1913)]. *J. H. Roberts.*

Please see the note to p. 241, Tsuji.

Moise, Edwin E. A note on the pseudo-arc. Trans. Amer. Math. Soc. 67, 57–58 (1949).

This is a short proof that the pseudo-arc (defined by the author) is homogeneous. A proof has previously been given by R. H. Bing [Duke Math. J. 15, 729–742 (1948); these Rev. 10, 261]. *J. H. Roberts* (Durham, N. C.).

Scott, W. R. Some elementary topological properties of essential maximal model continua. Bull. Amer. Math. Soc. 55, 963–968 (1949).

If $T: z=t(w)$ is a continuous plane transformation from a bounded connected open set D of the w-plane into a bounded set of the z-plane, if the essential multiplicity $k(z)$ is bounded and continuous, if A is any set of the z-plane and $E(A)$ the set of points covered by all essential maximal model continua of points of A, then, if A is closed, $E(A)$ is closed; if A is connected, $E(A)$ has a finite number of components. *L. Cesari* (Madison, Wis.).

GEOMETRY

***Banning, J. On the foundations of geometry.** Handelingen van het XXXI^e Nederlands Natuur- en Geneeskundig Congres, pp. 83–85, Haarlem, 1949. (Dutch)

As axioms for projective geometry the author proposes the simple properties of incidence along with the following self-dual form of Pappus's theorem. If two of the three diagonal points of a simple hexagon are respectively incident with two of the three diagonal lines, then the remaining diagonal point is incident with the remaining diagonal line. In the notation of F. Levi [Geometrische Konfigurationen, Hirzel, Leipzig, 1929, p. 108], where A_i, B_j, C_k are collinear whenever $i+j+k\equiv 0 \pmod 3$, the hexagon is $A_1B_1A_2B_2A_3B_3$, the diagonal points are C_1, C_2, C_3, and the diagonal lines are A_2B_3, A_3B_1, A_1B_2. [The author's own notation is less systematic. This hexagonal aspect of the Pappus configuration was also noticed by J. M. Feld, Amer. Math. Monthly 43, 549–555 (1936).] *H. S. M. Coxeter* (Toronto, Ont.).

***Loonstra, F. Structure and metric.** Handelingen van het XXXI^e Nederlands Natuur- en Geneeskundig Congres, pp. 91–93, Haarlem, 1949. (Dutch)

This provides a neat summary of Menger's paper on the foundations of projective and affine geometry [Ann. of Math. (2) 37, 456–482 (1936)] in which joining and meeting are expressed as addition and multiplication. *H. S. M. Coxeter* (Toronto, Ont.).

Davatz, W. Beiträge zum axiomatischen Aufbau der Geometrie. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A., no. 6, 157–190 (1939).

The author proposes a few small changes in the system of axioms for Euclidean geometry given by Hilbert [Grundlagen der Geometrie, 7th ed., Teubner, Leipzig, 1930]. He establishes the independence of the amended axioms by describing "geometries" in which each axiom in turn is denied. *H. S. M. Coxeter* (Toronto, Ont.).

✱Kaufmann, Karl. Gewebetheoretische Untersuchungen zur Axiomatik der dreidimensionalen affinen Geometrie. Thesis, Eidgenössische Technische Hochschule in Zürich, 1949. iii+41 pp.

This thesis treats three-dimensional affine geometry axiomatically by web-theoretic considerations. Point and plane are primitive, with the latter falling into four pairwise mutually exclusive classes. Lines are point sets common to two planes. It is assumed that (1) through each point there passes exactly one plane from each class, and (2) any three planes, each two from different classes, have exactly one point in common. These are the axioms of a web. If a_1, a_2 and b_1, b_2 are two pairs of skew lines, it is assumed that any three of the intersections $a_i \times b_j$ imply the fourth. The skew quadrangle of web lines is found to play an important role in the theory. The author shows how a coordinate system may be defined, whose elements belong to a noncommutative field. Commutativity is found to be equivalent to the existence of a certain ruled quadric.

L. M. Blumenthal (Columbia, Mo.).

✱Steiner, Jacob. Geometrical Constructions with a Ruler, Given a Fixed Circle with Its Center. Translated from the First German Edition (1833) by Marion Elizabeth Stark. Edited with an Introduction and Notes by Raymond Clare Archibald. Scripta Mathematica Studies, no. 4. Scripta Mathematica, New York, N. Y., 1950. iii+88 pp. (2 plates). $2.00.

Reprinted from Scripta Math. 14, 187–264 (1948); these Rev. 10, 562.

Dolaptschijew, Bl. Extreme bei der Bildung der Euklidisch-konvexen Polygone und Dreikantspolyeder. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 39, 1–56 (1943). (Bulgarian. German summary)

The author proves 18 theorems concerning the number and types of region into which n lines in general position divide the Euclidean plane, or n planes in general position divide Euclidean 3-space. Typical results are: four lines in general position bound one closed and one open 4-sided region in the plane, but five lines in general position do not in general bound any closed or open 5-sided region. Six planes in general position bound at least one and at most three closed 6-faced regions, and at most six and at least four open 6-faced regions; the total number of closed or open 6-faced regions which they bound is at most seven; the three closed 6-faced regions when they exist lie in a tetrahedron. Seven planes in general position do not in general bound any 7-faced region either closed or open.

O. Frink (State College, Pa.).

Trevisan, Giorgio. Una condizione di allineamento per gli insiemi finiti di punti del piano euclideo. Rend. Sem. Mat. Univ. Padova 18, 258–261 (1949).

The author rediscovers the following theorem [cf. H. S. M. Coxeter, Amer. Math. Monthly 55, 26–28 (1948); these Rev. 9, 458]. If a set S of a finite number of points of the Euclidean plane has the property that the line joining any two points of S contains a third one then S lies on one line. The proof seems to be new.

L. Fejes Tóth (Budapest).

Vakselj, Anton. Contributions à la géométrie du triangle. Akad. Ljubljani. Mat.-Prirodoslov. Razred. Mat. Odsek. Razprave 3, 43–75 (1947). (Slovenian. French summary)

Definition: a point of the plane of a triangle is said to be a symmetric (i.e., remarkable) point of that triangle if its coordinates $x_1:x_2:x_3$ are rational entire functions of the sines and cosines of the corresponding angles α, β, γ, if they are also such functions of the remaining two angles, and, in addition, are symmetric with respect to those two angles: $x_1=f(k\alpha;\, k\beta, k\gamma)=f(k\alpha;\, k\gamma, k\beta)$, $x_2=f(k\beta;\, k\gamma, k\alpha)$, $x_3=f(k\gamma;\, k\alpha, k\beta)$. In his calculations the author takes $\alpha+\beta+\gamma=\lambda$, where λ is an arbitrary constant. In the results obtained the particular values π and $\frac{1}{2}\pi$ are substituted for λ.

The author considers the two basic triads of points

$$(1)\qquad (\cos\alpha),\quad (\sin\alpha),\quad (\cos\beta\cos\gamma);$$
$$(2)\qquad (\cos^2\alpha),\quad (\cos\alpha\sin\alpha),\quad (\sin^2\alpha).$$

The points $(x)=p(\cos\alpha)+q(\sin\alpha)+r(\cos\beta\cos\gamma)$, where p, q, r are entire rational functions of $\sin\alpha\sin\beta\sin\gamma$, $\cos\alpha\cos\beta\cos\gamma$, $\cos\lambda$, $\sin\lambda$, form a subset M_1 of the total set M of symmetric points of the triangle. If $\alpha_k=k\alpha, \cdots, \lambda_k=k\lambda$, the triad of points $(\cos\alpha_k)$, $(\sin\alpha_k)$, $(\cos\beta_k\cos\gamma_k)$ may likewise be used to obtain a subset M_k of the set M. The basic triad (2) may be treated like the triad (1).

The author proves the theorem: any point of the set of points M belongs to a subset M_l obtained in this fashion, where l is a properly chosen rational number. The results obtained in the first part of the paper are used in the second part to define two kinds of special triangles: the opposite of an isosceles triangle, and quasi-similar triangles.

N. A. Court (Norman, Okla.).

Arvesen, Ole Peder. Sur les triangles de Poncelet. Norske Vid. Selsk. Forh., Trondhjem 20, no. 24, 92–95 (1948).

✱Duparc, H. J. A. Some applications of Casey's theorem. Handelingen van het XXXI^e Nederlands Natuur- en Geneeskundig Congres, pp. 85–87, Haarlem, 1949. (Dutch)

Herrmann, Aloys. Quelques théorèmes sur la géométrie du triangle. C. R. Acad. Sci. Paris 229, 1055–1056 (1949).

Thébault, Victor. Sur le quadrangle inscrit à un cercle. Math. Gaz. 33, 116–120 (1949).

Thébault, Victor. Sphères associées à un polygone gauche dont les sommets sont cosphériques. C. R. Acad. Sci. Paris 230, 271–273 (1950).

Thébault, Victor. Sur des coniques associées à un triangle. Ann. Soc. Sci. Bruxelles. Ser. I. 63, 74–80 (1949).

Thébault, Victor. On the Feuerbach points. Amer. Math. Monthly 56, 546–547 (1949).

Sandham, H. F. A generalization of Feuerbach's theorem. Amer. Math. Monthly 56, 620–622 (1949).

Droussent, L. A propos du théorème de Feuerbach. II. Propriétés des quadrilatères $ABCA''B'C'$, $ABCA'B''C'$, $ABCA'B'C''$. Mathesis 58, 230–247 (1949).

For part I cf. the same vol., 164–171 (1949); these Rev. 11, 125.

Droussent, Lucien. Coniques inscrites dont les foyers sont situés sur l'ellipse de Steiner inscrite. Bull. Soc. Roy. Sci. Liége 18, 312–332 (1949).

Court, Nathan Altshiller. Sur les cercles polaires des faces d'un tétraèdre. Mathesis 58, 222–224 (1949).

Gambier, B. Sur les tétraèdres dont certaines bihauteurs se recontrent. Bull. Soc. Math. France **77**, 139–140 (1949).

Blanchard, René. Sphères généralisées de Lucas. Mathesis **58**, 330–336 (1950).

Goormaghtigh, R. On pedal and antipedal triangles. Math. Gaz. **33**, 105–107 (1949).

Goormaghtigh, R. Sur une généralisation du théorème de Pollock. Mathesis **58**, 325–329 (1950).

Goormaghtigh, R. Sur les isoptiques de deux hypocycloïdes à trois rebroussements. Mathesis **58**, 285–288 (1950).

Wunderlich, Walter. Über die Schleppkurven des Kreises. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 155–173 (1948).

The circular tractrix is an orthogonal trajectory of the system of circles of equal radius l whose centers lie on a fixed circle of radius a [Morley, Ann. of Math. (2) **1**, 21–30 (1899)]. The ordinary tractrix arises as the limiting case when a tends to infinity. The circular tractrix is said to be of parabolic type if $a=l$, elliptic if $a<l$, hyperbolic if $a>l$. The parabolic type is Cotes's "tractrix complicata," the inverse of the involute of a circle. In every other case the inverse (with respect to a concentric circle) is similar to the original curve. Such a curve can be derived by stereographic projection from a spherical tractrix, which is an orthogonal trajectory of the sections of a sphere by the tangent planes of a cone of revolution whose axis joins the center of the sphere to the pole of the stereographic projection. The parabolic type arises when the apex of the cone lies on the sphere, the elliptic when it is inside, the hyperbolic when it is outside. The circular tractrix of elliptic or hyperbolic type is found to be the Klein-Poincaré model of the involute of a circle in the elliptic or hyperbolic plane. The projection of a spherical tractrix from any point on the axis of the cone is found to be the Beltrami-Cayley-Klein model of a circular tractrix in the elliptic or hyperbolic plane, according as the center of projection is inside or outside the sphere.

H. S. M. Coxeter (Toronto, Ont.).

✶Kavafian, K. K. Étude élémentaire de la quadrice et quelques applications des coordonnées bipolaires. Actualités Sci. Ind. no. 1069, Hermann et Cie., Paris, 1949. 60 pp.

A quadrice is the locus of points in the plane whose distances from two fixed points A, B, taken on two fixed lines D, D', are equal to the lengths of the segments Au, Bv determined on D, D' by a variable line Suv revolving about a fixed point S. The points A, B are the foci and the line AB an axis of symmetry of the curve. The quadrice is bicircular, i.e., the cyclic points at infinity are double points of the curve. The quadrice may also have a real double point, but no singularities of a higher order. The quadrice is, in general, of order eight. The ovals of Descartes and of Cassini are particular cases of the quadrice, and the curve may degenerate into a circle, an ellipse, or an hyperbola. A circle may cut a quadrice in eight points; the inverse of the quadrice is also a curve of the eighth degree. The author derives some properties of the tangent and the normal, considers pencils of confocal quadrices, their envelope, etc.

The reading of the booklet is facilitated by fifteen figures in the text. However, these figures do not include a single picture of the quadrice, in spite of the fact that the author describes a linkage, named homograph, with which the curve can be drawn by continuous motion.

N. A. Court (Norman, Okla.).

Deaux, R. Sur la cubique de Mac Cay. Mathesis **58**, 225–230 (1949).

Ambasankar, J. A. Grassmann cubic and Wallace lines. Math. Student **16** (1948), 8–17 (1949).

Mandan, Ram. Segre's quartic locus. Bull. Calcutta Math. Soc. **41**, 140–142 (1949).

Beth, H. J. E. On geometries. Nieuw Tijdschr. Wiskunde **34**, 111–127, 230–244 (1947); **36**, 219–224, 279–288, 332–351 (1949); **37**, 8–21 (1949). (Dutch)

After an historical sketch of elementary geometry from the Babylonians and Egyptians through Euclid and Apollonius to Monge, the author summarizes the essential ideas of projective geometry. He uses Steiner's construction for a conic, but his approach to cross ratio and homogeneous coordinates is metrical. He gives a synthetic account of congruences and complexes of lines, constructing a linear complex with the aid of two projectively related flat pencils in distinct planes. He gives the following construction for the polar plane of a given point P with respect to the linear complex determined by five given lines $a_1, \cdots, a_5$: let the transversals from P to a_1, a_2, a_3, a_4 be l and m, meeting the plane Pa_5 in L and M; then the polar plane of P joins the point $LM\cdot a_5$ to the line $Pl\cdot Pm$. He then considers four-dimensional geometry (projective, affine and Euclidean), describing the figure of five associated planes and the various kinds of parallelism and orthogonality. After a brief return to the Euclidean plane for a discussion of roulettes, he describes some simple results in non-Euclidean geometry (mostly hyperbolic). *H. S. M. Coxeter* (Toronto, Ont.).

Coxeter, H. S. M. Projective geometry. Math. Mag. **23**, 79–97 (1949).

This paper is in the main expository, but contains an apparently new proof of the well-known theorem that every elliptic projectivity on a line is direct, and a somewhat unusual treatment of polarities in the real projective plane.

J. A. Todd (Cambridge, England).

Vlahavas, G. N. Une famille de droites concourantes. Elemente der Math. **4**, 88–89 (1949).

If P, Q are the centroids of the given sets of points p, q, and O is the centroid of the points $p+q=n$, the author proves, once again, that the points P, O, Q are collinear and that we have, both in magnitude and in sign, $QO:OP=p:q$. For given values of p and q the number of lines POQ that may be obtained is $N(p,q)={}_pC_n={}_qC_n$; hence if n remains fixed and p takes on all possible values, the number of lines obtained, all passing through O, is $\sum N(p,q)=\frac{1}{2}\sum_{p=1}^{n-1}{}_pC_n=2^{n-1}-1$. The author applies these considerations to several configurations and readily obtains his results, most of them quite familiar. *N. A. Court.*

Kommerell, Karl. Mehrfach ausgeartete Desarguessche Konfigurationen. Math. Z. **52**, 472–482 (1949).

There is a polarity (discovered by von Staudt) for which the ten points of the Desargues configuration are the poles of its ten lines. The configuration is said to be degenerate if one of the points lies on its corresponding line, in which case

the polarity, being hyperbolic, determines a conic α. The author shows that the configuration may be simply, doubly or triply degenerate, but no more. In terms of the hyperbolic geometry with α as absolute, the triply degenerate configuration consists of the vertices, sides, altitudes and center of a triply asymptotic triangle, along with the polars and poles of these four points and six lines. The altitudes decompose the triangle into six simply asymptotic right triangles. Reflections in the sides of such a right triangle generate Klein's modular group $A^2=B^2=C^2=(AB)^2=(AC)^3=1$, of which the finite subgroup $A^2=C^2=(AC)^3=1$ is the collineation group of the triply degenerate Desargues configuration, A and C being harmonic homologies.

H. S. M. Coxeter (Toronto, Ont.).

Blanchard, René. Sur une configuration de cinq droites. Mathesis 58, 306–309 (1950).

Rodeja F., E. G-. Note sur les ellipses d'aire minimum appartenant à un faisceau de coniques. Nieuw Tijdschr. Wiskunde 37, 166–172 (1949).

The general conic through four fixed points is $f_1-\lambda f_2=0$. The product of the two semi-axes is stationary if λ satisfies a certain cubic equation whose roots are all real. If the four points form a convex quadrangle, one of these roots yields the ellipse of minimum area [Bottema, same Tijdschr. **35**, 126–140 (1947); these Rev. **9**, 303]. The author points out that the other two roots (and all three when the points do not form a convex quadrangle) have likewise a simple affine interpretation: they yield hyperbolas for which the area of the triangle formed by a tangent and the asymptotes is a relative maximum. *H. S. M. Coxeter* (Toronto, Ont.).

Locher-Ernst, L. Das Imaginäre in der Geometrie. Elemente der Math. **4**, 97–105, 121–128 (1949).

Let M be the center of a given elliptic involution of points (I), and let N, N' be that pair of (I) for which we have, in magnitude and in sign, $NM=MN'$. The two segments MN, MN' surmounted by arrows going from M to N, N', respectively, are taken as the symbolic representation of the conjugate imaginary points constituted by (I). This is a compact metrical version of Staudt's projective definition of two such points. The definition of an imaginary line in the plane is derived from the one of the point. The author uses this representation to prove some of the incidence relations of imaginary elements in the plane. But the main advantage claimed for the innovation is that it makes the imaginary elements more intuitive and that it lends itself to picturing those elements graphically. The author offers a number of such graphs. The imaginary elements in space are treated in a like manner.

N. A. Court (Norman, Okla.).

Belgodère, Paul. Les géométries de figures orientées. II. Gaz. Mat., Lisboa **10**, no. 40, 11–13 (1949).

For part I cf. the same vol., no. 39, 2–4 (1949); these Rev. **10**, 618.

Cenov, Iv. Free vectors and their application in analytic geometry. Sbornik Bulgar. Akad. Nauk **38**, 1–128 (1942). (Bulgarian)

Piel, Carl. Die Clifford'schen Parallelen und die Clifford'sche Fläche. Math.-Phys. Semesterber. **1**, 99–108 (1949).

This is an elementary exposition of parallelism in elliptic space, using canonical homogeneous coordinates. It begins with an historical sketch, giving due credit to Riemann, Clifford, Klein, Bonnesen, Liebmann, Bonola and Rosemann, though there is no mention of Study or Cartan. After establishing the transitivity of Clifford parallelism, the author gives a reference to Gauss which is misleading, for of course Gauss considered only hyperbolic geometry, not elliptic. *H. S. M. Coxeter* (Toronto, Ont.).

★**Roever, William Henry. The Axonometric Method of Descriptive Geometry.** St. Louis, Mo., 1949. vi+75 pp.

Palm, F. W. Über den Perspektivumriss einer allgemeinen Schraubfläche. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 63–78 (1949).

A general helicoid is a surface generated by the helical motion of a plane curve (the meridian) about an axis located in its plane. The author gives a construction for the contour of a helicoid in central projection. He determines the points on the surface whose images lie on the contour by introducing a special "helical mapping." *E. Lukacs.*

Laub, Josef. Bemerkungen zu einer Dreiecksaufgabe, die in der Geodäsie eine Rolle spielt. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 7, i+14 pp. (1947).

Given three straight lines g_1, g_2, g_3 and a triangle with vertices A_1, A_2, A_3, the author discusses the problem of finding triangles congruent to $A_1A_2A_3$ such that the point A_i lies on the line g_i $(i=1, 2, 3)$. This problem occurs in surveying in connection with resection. *E. Lukacs.*

Schaller, H. Zum räumlichen Rückwärtseinschneiden. Monatsh. Math. **53**, 184–186 (1949).

The paper deals with a graphical solution of the problem of resection in space. The author claims that his solution is exact and contrasts it in this respect with earlier solutions such as the one obtained by S. Finsterwalder [Jber. Deutsch. Math.-Verein. **6**, Heft 2, 1–90 (1899)]. The author obtains his solution by intersecting a quartic curve with a circle. Essentially similar solutions of this problem are well known and may be found in standard texts [Th. Schmid, Darstellende Geometrie, v. 2, 2d ed., de Gruyter, Berlin, 1923, pp. 307–309]. The reviewer does not see any property of this solution which would justify calling it exact; a graphical solution would normally be called exact if it can be obtained by ruler and compass. Any other use of the term "exact solution" would require a definition which the author fails to give. *E. Lukacs* (Washington, D. C.).

Rössler, Fred. Über verallgemeinerte Reliefperspektiven. Monatsh. Math. **53**, 211–220 (1949).

The author continues his study of nonlinear relief perspectives [Z. Angew. Math. Mech. **28**, 311–316 (1948); these Rev. **10**, 394]. In the present paper he replaces the picture plane and the vanishing plane by the following. (1) Two concentric spheres. The center of projection is the center of the spheres. (2) Two similar quadric surfaces of revolution with common axis and one common focus. The common focus is the center of similitude and also the center of projection. (3) Two congruent quadric cones of revolution with common axis. The center of projection is a point on the common axis which does not separate the vertices. (4) Two congruent quadric cones with common focal axis. The center of projection is a point on the focal axis which does not separate the vertices. *E. Lukacs.*

Reuschel, Arnulf. Eine einfache Berechnung der Mantelfläche eines Drehkegelhufes. Elemente der Math. **4**, 73–78, 133–138 (1949).

Convex Domains, Extremal Problems

Fejes Tóth, László. Extremum properties of the regular polyhedra. Canadian J. Math. **2**, 22–31 (1950).

Work on the determination of the polyhedra of maximum volume for a given surface and a given number of faces is summarized. The work of Steiner, Lindelöf, Minkowski and Steinitz showed that the regular polyhedra of 4 and 6 faces were maximum, but that for 8 and 20 faces they were not maximum. The reviewer showed that the regular dodecahedron was maximum by a method which gave the maximum polyhedra for 4, 6 and 12 faces in one simple demonstration [Tôhoku Math. J. **40**, 226–236 (1935)]. The demonstration has been amplified in other papers by the author. In the present paper it is shown that, if V is the volume of a convex polyhedron of f faces, v vertices and e edges, r is the radius of the insphere, R is the radius of the circumsphere and F is the surface, then

$$V \geqq (e/3) \sin(\pi f/e)\{\tan^2(\pi f/2e)\tan^2(\pi v/2e)-1\}r^3$$

and

$$V \leqq (2e/3)\cos^2(\pi f/2e)\cot(\pi v/2e) \times\{1-\cot^2(\pi f/2e)\cot^2(\pi v/2e)\}R^3.$$

Equality holds in both formulas for all five of the regular polyhedra. It is conjectured that for all polyhedra

$$F^3/V^2 \geqq 9e\sin(2\pi/p)\{\tan^2(\pi/p)\tan^2(\pi/q)-1\},$$

where p is the average number of edges around a face and q is the average number of edges radiating from a vertex.

M. Goldberg (Washington, D. C.).

Fejes Tóth, L. On the total length of the edges of a polyhedron. Norske Vid. Selsk. Forh., Trondhjem **21**, no. 8, 32–34 (1948).

It is conjectured that the sum L of the lengths of the edges of a convex polyhedron containing a sphere of diameter D is given by $L \geqq 12D$. This relation is proved for polyhedra of equiareal faces. The cube satisfies the equality condition. For all convex polyhedra, it is proved that $L > 10D$. For polyhedra of triangular faces, it is proved that $L > 14D$. *M. Goldberg* (Washington, D. C.).

Kneser, Martin. Eibereiche mit geraden Schwerlinien. Math.-Phys. Semesterber. **1**, 97–98 (1949).

If a closed convex curve in the plane has the property that the loci of the midpoints of parallel chords are straight lines, then the curve is an ellipse. The author gives a simple proof of this well-known theorem without making differentiability assumptions. *W. Fenchel* (Los Angeles, Calif.).

Dinghas, Alexander. Einfacher Beweis der isoperimetrischen Eigenschaft der Kugel im euklidischen Raum von n Dimensionen. Math. Nachr. **2**, 107–113 (1949).

Dinghas, Alexander. Einfacher Beweis der isoperimetrischen Eigenschaft der Kugel in Riemannschen Räumen konstanter Krümmung. Math. Nachr. **2**, 148–162 (1949).

The papers will be cited as I, II, respectively. Let R_n denote an n-space which is either Euclidean, hyperbolic or spherical. Only point sets are considered that are bounded in the first two cases or lie in an open hemisphere in the last one, respectively. The proof in II of the isoperimetric inequality for closed sets in any R_n runs parallel to the Euclidean proof in I. A "body" $\mathfrak{K}$ is a non-empty connected Lebesgue measurable point set in R_n. Let $V(\mathfrak{K})$ denote the volume of $\mathfrak{K}$; $\mathfrak{S}$ the sphere in R_n with $V(\mathfrak{S})=V(\mathfrak{K})$; $\mathfrak{K}_h$ the set of those points whose distance from $\mathfrak{K}$ is at most h ($h>0$; h sufficiently small in the spherical case). Let $\overline{\mathfrak{K}}$ denote the body obtained by symmetrizing $\mathfrak{K}$ with respect to some $(n-1)$-dimensional plane E. Thus (1) $V(\overline{\mathfrak{K}})=V(\mathfrak{K})=V(\mathfrak{S})$.

The following observation is systematically used in II. For a given plane E, the space R_n can be mapped on a Euclidean model so that volume and symmetrization with respect to E are preserved. Since $\overline{\mathfrak{K}}_h \subset \overline{\mathfrak{K}_h}$ in the Euclidean case, this remark yields (2) $V(\overline{\mathfrak{K}}_h) \leqq V(\mathfrak{K}_h)$ for any R_n [cf. (1)]. In the same way, "Hadwiger's lemma" is proved. Let $\mathfrak{L}$ be another body in R_n. Then

$$V(\overline{\mathfrak{K}}\cdot\overline{\mathfrak{L}})=V(\mathfrak{K}\cdot\mathfrak{L})+V(\overline{\mathfrak{K}-\mathfrak{L}}\cdot\overline{\mathfrak{L}-\mathfrak{K}}) \tag{3}$$

[cf. Elemente der Math. **3**, 25–38 (1948); these Rev. **9**, 526].

Let $\mathfrak{K}^*$ run through those bodies that are obtained by symmetrizing $\mathfrak{K}$ a finite number of times. Using (3), the author proves $\sup V(\mathfrak{K}^*\mathfrak{S})=V(\mathfrak{S})$. Hence, to every k with $0<k<h$ there is a $\mathfrak{K}^*$ such that $\mathfrak{S}_{h-k}\subset(\mathfrak{K}^*\mathfrak{S})_h$. Applying (2) and letting $k\to 0$, he obtains (4) $V(\mathfrak{K}_h)\geqq V(\mathfrak{S}_h)$. From now on, let $\mathfrak{K}$ be closed. The Minkowski surface of $\mathfrak{K}$ is defined by $O(\mathfrak{K})=\liminf_{h\to 0}(V(\mathfrak{K}_h)-V(\mathfrak{K}))/h$. Thus (4) implies the isoperimetric inequality (5) $O(\mathfrak{K})\geqq O(\mathfrak{S})$. The equality sign is discussed in two cases. (a) Given a straight line a; the union of the convex closures of the intersections of $\mathfrak{K}$ with the straight lines parallel to a form a closed body $\mathfrak{K}_a$ [a slightly different formulation is necessary in II]. Suppose $\mathfrak{K}_a-\mathfrak{K}$ possesses interior points for some a. Then symmetrization with respect to a plane normal to a yields an improvement of (2) which implies (6) $O(\mathfrak{K})>O(\overline{\mathfrak{K}})$. Since $O(\overline{\mathfrak{K}})\geqq O(\mathfrak{S})$ from (5) and (1), we obtain (7) $O(\mathfrak{K})>O(\mathfrak{S})$. (b) Let $\mathfrak{K}_a-\mathfrak{K}$ be empty for every a, i.e., let $\mathfrak{K}$ be convex. In this case, a new and very simple proof of (7) for $\mathfrak{K}\neq\mathfrak{S}$ is given. Move $\mathfrak{S}$ so that $V(\mathfrak{K}\mathfrak{S})$ is a maximum. From (3) it follows that $V(\overline{\mathfrak{K}}\mathfrak{S})>V(\mathfrak{K}\mathfrak{S})$ for some E through the center of $\mathfrak{S}$. Hence $\mathfrak{K}$ and $\overline{\mathfrak{K}}$ are not congruent, i.e., $\mathfrak{K}$ is not symmetric with respect to any plane parallel to E. This again implies (6) [proof in II] and therefore (7). [In the spherical case, this proof has to be modified.] *P. Scherk*.

v. Sz. Nagy, Gyula. Schwerpunkt von konvexen Kurven und von konvexen Flächen. Portugaliae Math. **8**, 17–22 (1949).

The author proves that $d<4\delta<3d$ for a convex curve, and $d<6\delta<5d$ for a convex surface, where d is the distance apart of two parallel lines, or planes, of support and where δ is the distance of one (or the other) of them from the centre of gravity of a homogeneous mass distribution on the curve or surface. The general case is deduced from that of the perimeter of a triangle and that of the surface of a tetrahedron. The constants occurring in the inequalities are best possible, aready in the above special cases.

L. C. Young (Madison, Wis.).

Sz.-Nagy, Gyula. Zentralsymmetrisierung konvexer Körper. Publ. Math. Debrecen **1**, 29–32 (1949).

Démonstration élémentaire du fait qu'étant donné un corps convexe, le corps obtenu par symétrisation de centre un point fixe O, qui est l'homothétique du corps vectoriel

dans le rapport $\frac{1}{2}$, peut être défini aussi bien au moyen des plans d'appui, qu'au moyen des symétrisations des largeurs dans toutes les directions. Application: la somme des rayons de courbure d'un orbiforme plan aux deux extrémités d'un même diamètre est égale à la largeur de l'orbiforme.

J. Favard (Paris).

Kelly, Paul J. On Minkowski bodies of constant width. Bull. Amer. Math. Soc. **55**, 1147–1150 (1949).

A bounded subset of a metric space is called entire if the addition of any point to the set increases its diameter. In Euclidean n-space an entire set is a convex body of constant width, and conversely. [For references see T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, p. 129.] The author shows, using concepts recently introduced by H. Busemann, that this theorem also holds in a Minkowski n-space. *W. Fenchel.*

Misra, D. C. A property of closed regions. Bull. Calcutta Math. Soc. **41**, 83–85 (1949).

Fallacious. The inequality $p(\alpha)+p(\alpha+\pi)\leqq 2r$ of page 85 is subject to the assumption that the tangents at α and $\alpha+\pi$ are parallel. *L. C. Young* (Madison, Wis.).

Matos Peixoto, Mauricio. On convexity. Anais Acad. Brasil. Ci. **21**, 291–302 (1949).

To define generalized convex regions in the plane, the author augments any family $\{F(x)\}$ of continuous functions $F(x)$, such that there is a unique member of $\{F(x)\}$ through each pair of points (x_1, y_1), (x_2, y_2) with $x_1\neq x_2$, by the set of vertical straight lines; the resulting family is denoted by $\{F(x)\}^*$. (The families $\{F(x)\}$ are the families used in defining generalized convex functions.) A set Δ of points in the plane is said to be convex relative to $\{F(x)\}^*$ provided that, for each pair of distinct points A and B of Δ, the arc between A and B of the curve of $\{F(x)\}^*$ through A and B lies in Δ. The author now discusses generalized convex sets Δ and their boundaries. Thus if such a Δ is closed and bounded and has interior points, then the boundary of Δ is a Jordan curve, through each point of which there passes at least one curve of $\{F(x)\}^*$ which is a supporting curve of Δ. The closed convex hull of a set relative to $\{F(x)\}^*$ is also discussed.

We note that the analysis might equally well have been made relative to any family $\{C\}$ of continuous curves C in a domain D, each of which separates D into two components, and such that for each pair of distinct points A and B of D there is a unique member of $\{C\}$ through A and B. *E. F. Beckenbach* (Los Angeles, Calif.).

Haupt, Otto. Über einige affingeometrische Ovalsätze in der direkten Infinitesimalgeometrie. Math. Z. **51**, 635–657 (1949).

In a previous paper [Abh. Math. Sem. Hansischen Univ. **15**, 130–164 (1943); these Rev. **7**, 474] the author has generalized certain classical theorems on the intersections of an oval with conic sections, replacing the oval by an arbitrary arc or curve and the conic sections by a family of curves called order characteristics having certain postulated properties. In this earlier topological approach, conditions on the arc and order characteristics were stated "in the small," that is, in terms of an arbitrarily small neighborhood of a point, but conditions "in the infinitely small," that is, conditions involving limiting behavior of the order characteristics at a point of the arc, were largely avoided. In the present paper conditions on the limiting behavior of order characteristics at a point of the arc, i.e., "in the infinitely small," are given which are sufficient to insure the validity of the theorems in the earlier paper. The approach is that of direct infinitesimal geometry in that the assumptions regarding the limiting behavior of order characteristics are stated explicitly in geometric form rather than being disguised as assumptions of differentiability as in classical differential geometry. *S. B. Jackson.*

Algebraic Geometry

✶Walker, Robert J. Algebraic Curves. Princeton Mathematical Series, vol. 13. Princeton University Press, Princeton, N. J., 1950. x+201 pp. **$4.00.**

Ce livre est un représentant d'une tendance récente [cf. Hodge et Pedoe, Methods of Algebraic Geometry, v. 1, Cambridge University Press; Macmillan, New York, 1947; Segre, Lezioni di Geometria Moderna, v. 1, Zanichelli, Bologna, 1948; ces Rev. **10**, 396, 729], et constitue une introduction à la géométrie algébrique moderne, destinée aux débutants. Le traitement en est purement algébrique, et la théorie "transcendante" des courbes algébriques (surfaces de Riemann, homologie, fonctions thêta) n'y est pas abordée. D'autre part, et pour éviter de compliquer l'exposé par l'introduction de techniques algébriques supplémentaires, l'auteur s'est borné au cas d'un corps de base algébriquement clos et de caractéristique nulle. Après deux chapitres préliminaires, l'un portant principalement sur les anneaux de polynomes, et l'autre sur les espaces projectifs, le chapitre III aborde l'étude des courbes algébriques planes. On se heurte bientôt à l'obstacle des points singuliers, et on se rend vite compte de la nécessité de techniques nouvelles pour étudier les points multiples à tangentes non distinctes: celle des transformations quadratiques est introduite à la fin du chapitre III; celle des séries formelles fait l'objet du chapitre IV, et permet de démontrer, en toute généralité, le théorème de Bézout et les formules de Plücker. Le chapitre V est consacré aux transformations birationnelles et rationnelles d'une courbe, et la technique nécessaire de théorie des corps y est introduite; on y montre qu'une correspondance birationnelle entre deux courbes définit une correspondance biunivoque entre leurs places (qui ont été définies au chapitre IV comme classes de paramétrisations équivalentes d'une courbe par des séries formelles), ce qui introduit les valuations du corps des fonctions rationnelles sur une courbe; les courbes de l'espace sont introduites comme transformées birationnelles des courbes planes, puis caractérisées par un idéal premier de dimension 1 d'un anneau de polynomes. Enfin le chapitre VI est consacré aux séries linéaires et aux transformations rationnelles qui leur sont associées (et qui donnent rapidement la réduction des singularités); le genre est défini au moyen de la série canonique; le théorème de Riemann-Roch est démontré par double récurrence (sur la dimension, puis sur l'indice de spécialité), et est suivi de ses conséquences classiques (Brill-Noether, Clifford, Weierstrass, classification des courbes en hyperelliptiques et non hyperelliptiques).

Les principaux théorèmes sont illustrés par de nombreux exemples les uns dans le texte, les autres en exercices. Parmi les exemples du texte, celui des cubiques planes est souvent utilisé. Parmi les exercices se trouvent aussi quelques généralisations aux variétés de dimension supérieure. Il semble que l'auteur s'est volontairement abstenu d'écrire un ouvrage

"pur," où dominerait un des nombreux points de vue sous lesquels on peut considérer les courbes algébriques; et que son double but a été, d'une part de donner au lecteur, avec toute la précision maintenant exigée, l'accès de plain-pied aux mémoires des géomètres classiques, et d'autre part de l'initier, sur des exemples simples, aux diverses techniques algébriques des géomètres modernes. *P. Samuel.*

Amato, Vincenzo. Le curve algebriche nella teoria delle equazioni secondo Galois. Boll. Un. Mat. Ital. (3) **4**, 104–109 (1949).

The equation of an irreducible algebraic curve is considered from the point of view of general Galois theory.

C. Chevalley (New York, N. Y.).

Room, T. G. Involutory unitary matrices of integers associated with certain geometric transformations. Quart. J. Math., Oxford Ser. **20**, 193–217 (1949).

Three systems are considered, leading to isomorphic "groupoids," or sets of operators such that the product of any odd number of them belongs to the set. They have also the properties that $A^{-1}=A$, $ABC=CBA$, for all operators of the set.

(I) The operations are the involutary self-transformations of a cubic curve by lines through a fixed point of it. Syzygetic systems of operations and corresponding points are considered, namely sets such that the points into which any point of the system is transformed by any operator of the system and the operator corresponding to this point also belong to the system. The n-fold syzygetic system generated by points whose elliptic parameters are $\phi_1, \cdots, \phi_n$ consists of all points whose parameters are $\sum_{i=1}^{n}\alpha_i\phi_i$, where $\alpha_1, \cdots, \alpha_n$ are integers and $\sum_{i=1}^{n}\alpha_i\equiv 1 \pmod 3$. The cases $n=1, 2$ are studied in detail.

(II) On a surface π of order $N+2$ in three dimensions with an $(N-1)$ple line v, if a is a rational curve of order m to which v is $(m-1)$-secant (and the pencil of plane cubics u residual to v consequently unisecant), the operation considered is the interchange of the two residual intersections with π of every line meeting v and a. Every groupoid of such operations is isomorphic with that of the operations (I) on each cubic u, corresponding to the points traced by the various curves a on u.

(III) If curves $c_1, \cdots, c_r$, or $\mathbf{c}$, with intersection matrix $\mathbf{\Gamma}$, form a base for algebraic curves on π, and are transformed by one of the operations (II) into $\mathbf{Tc}$, then $\mathbf{T\Gamma T'}=\mathbf{\Gamma}$; whence since $\mathbf{T}^2=\mathbf{I}$, $\mathbf{T\Gamma}$ is symmetrical. For an n-fold syzygetic system of curves a partial base is used consisting of u and $n+1$ curves $a_0, \cdots, a_n$ of the system (for $n=1$, the generator and its transform by the operation corresponding to itself, for $n=2$ the two generators and a curve which is the transform of either by the other). For these $\mathbf{\Gamma}$ has 0 in the first row and column, and the rest of each of these consists of 1's. The matrix $\mathbf{T}$ of the operation corresponding to a curve $\rho u+\sum_{i=0}^{n}\alpha_i a_i$ of the system is $\mathbf{Y}_\alpha=\rho\mathbf{U}+\sum_{i=0}^{n}\alpha_i\mathbf{A}_i$, where $\mathbf{U}$ consists of 1's throughout the first column, except in the first row, and 0's everywhere else, and $\mathbf{A}_i$ is the matrix of the transformation corresponding to a_i. (The conditions for $\rho u+\sum_{i=0}^{n}\alpha_i a_i$ to be a curve of the kind required are that $\sum_{i=0}^{n}\alpha_i=1$ and ρ is a quadratic function of the α's, depending on $\mathbf{\Gamma}$.) The matrices $\mathbf{Y}_\alpha$ satisfy also $\mathbf{Y}_\alpha\mathbf{Y}_\beta\mathbf{Y}_\gamma=\mathbf{Y}_{\alpha-\beta+\gamma}$ (where α indicates the row of coefficients $\alpha_1, \cdots, \alpha_n$). In a note added later the matrices considered here are interpreted as collineations in projective space of $n+2$ dimensions. They all prove to be harmonic inversions with respect to a line and an n-dimensional space, pole and polar with regard to a fixed quadrix (determined by $\mathbf{\Gamma}$), the lines all passing through a fixed point. *P. Du Val.*

Wylie, C. R., Jr. The uniqueness of a certain line involution. Bull. Amer. Math. Soc. **55**, 633–638 (1949).

The author considers a line involution in S_3 with the property that a general line and its image do not intersect, to which the author has associated, besides its order m, and the order i, of its invariant complex, the number k of lines of an arbitrary pencil which meet their images, and another numerical characteristic n associated with the mapping relation between the lines of S_3 and the points of a nonsingular V_4^2 in S_5. These four numbers are not independent, but satisfy the two relations $n-i=k$, $n+i=m+1$. In the present note the author establishes the fact that there is a unique involution associated with the set $m=n=2$, $i=k=1$, namely the involution whose point equivalent on V_4^2 is defined by the transversals of a general line l of S_5 and an S_3 contained in the S_4 which is tangent to V_4^2 at one of the two intersections of l and V_4^2. *M. Piazzolla-Beloch.*

Baldassarri, Mario. Sulle caratteristiche per le condizioni doppie e triple delle coniche. Rend. Sem. Mat. Univ. Padova **18**, 54–67 (1949).

Dans le but d'étudier ultérieurement de façon systématique les problèmes de contact de coniques avec une courbe algébrique, l'auteur détermine ici une base minima pour les conditions doubles ou triples. Le point de départ et les notations sont dans un mémoire de Severi sur les fondements de la géométrie énumérative [Ann. Mat. Pura Appl. (4) **19**, 153–242 (1940); ces Rev. **7**, 476]. On représente les coniques complètes (ponctuelle+tangentielle) par les points de la variété M_5 de S_{26} modèle projectif des couples point-hyperplan polaire par rapport à une surface de Veronese. Severi a déterminé sur M_5 la base minima des courbes et des variétés V_4 et donné quelques suggestions [loc. cit., § 66] concernant les variétés V_3 et V_2. L'auteur détermine la base minima pour ces variétés. Pour les variétés V_3, on obtient (1) les coniques passant par deux points; (2) les coniques passant par un point avec tangente donnée; (3) les coniques tangentes à deux droites données; (4) les coniques décomposées en deux droites conjuguées par rapport à une enveloppe de seconde classe; (5) les enveloppes décomposées en deux points conjugués par rapport à une conique donnée. Pour les variétés V_2 on obtient (A) les coniques passant par trois points; (B) les coniques d'un réseau contenant une droite double; (C) les coniques tangentes à une droite donnée et passant par un point avec tangente donnée; (D) les coniques décomposées en deux droites issues d'un point donné; (E) les enveloppes décomposées en deux points sur une droite donnée. Le premier de ces deux résultats est seul en accord avec la conjecture de Severi. Comme application, l'auteur détermine les deux caractéristiques suivantes:

$$\sigma=7A-9B+3C-3D+5E,$$
$$\beta=(5m-4n)\sigma+mD+(6m-4n)E;$$

la première est la condition pour qu'une conique se décompose et contienne une droite donnée; la seconde est la condition pour qu'une conique soit surosculatrice à une courbe algébrique d'ordre n et de classe m. *L. Gauthier.*

Niče, Vilim. Le foyer quadruple des courbes circulaires unicursales du 3ᵉ ordre et leurs faisceaux spéciaux. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **2**, 13–14 (1949).

L'auteur rattache diverses propriétés des courbes circulaires unicursales du troisième ordre, à la considération de celles-ci comme projections des sections planes d'un conoïde de Plücker sur un plan directeur. Il indique des constructions simples du foyer quadruple d'une cubique circulaire unicursale; envisage des faisceaux de telles courbes ayant en commun un point propre A, le point double S et le foyer quadruple F, et, au subjet de ces faisceaux, indique les propriétés suivantes. (a) Les asymptotes des courbes d'un même faisceau forment un faisceau dont le sommet, A_2, est situé sur la droite SA, à une distance de S égale à celle du point A au point A_1 où SA coupe le cercle de centre F et de rayon FA. (b) Étant donné un faisceau de strophoïdes ayant en commun le point double et le foyer quadruple, les asymptotes des courbes du faisceau forment un faisceau, tel que, le point double soit le milieu du segment déterminé par le sommet du faisceau et le foyer quadruple. Dans (a) comme dans (b), les enveloppes des tangentes aux courbes des faisceaux envisagés parallèles aux asymptotes sont des cercles. *P. Vincensini* (Marseille).

Niče, Vilim. Sur les courbes circulaires unicursales du 4ᵉ ordre à un point double à l'infini. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **2**, 15–16 (1949).

Ce travail fait suite au précédent relatif aux cubiques circulaires unicursales. Les courbes circulaires unicursales du quatrième ordre ayant un point double à l'infini y sont envisagées comme projections, sur le plan directeur, des sections d'un conoïde droit du quatrième ordre dont les génératrices à l'infini sont deux droites minima. Après avoir signalé diverses propriétés se rattachant au point de vue indiqué, l'auteur donne une construction simple du foyer quadruple de l'une quelconque des courbes considérées. Il fournit un moyen pour reconnaître si une courbe unicursale à point double à l'infini est circulaire; montre que toute courbe unicursale circulaire du quatrième ordre ayant un point double à l'infini est une cissoïde, et signale une transformation affine transformant toute courbe unicursale du quatrième ordre ayant un point double et deux points imaginaires à l'infini en une courbe circulaire de même espèce. De là résulte que toute courbe unicursale du quatrième ordre ayant un point double et deux points imaginaires à l'infini peut être regardée comme une cissoïde de ses asymptotes et de quatre courbes unicursales du troisième ordre ayant deux points imaginaires à l'infini, contenant deux à deux un des points doubles à distance finie de la courbe du quatrième ordre envisagée, et touchant cette dernière aux points imaginaires à l'infini.
P. Vincensini (Marseille).

Niče, Vilim. The quadruple focus of unicursal circular cubic curves and their special bundles. Rad Jugoslav. Akad. Znanosti i Umjetnosti **271**, 25–32 (1948). (Croatian)

Niče, Vilim. On circular quartic curves of degree zero with a double point at infinity. Rad Jugoslav. Akad. Znanosti i Umjetnosti **271**, 33–39 (1948). (Croatian)

Summaries are reviewed above.

Tanturri, Giuseppe. I fasci di curve piane algebriche con hessiana fissa. Atti Sem. Mat. Fis. Univ. Modena **3**, 210–220 (1949).

L'auteur dresse une classification complète des faisceaux composés de courbes non toutes réductibles, dont la hessienne est fixe. Un tel faisceau contient toujours trois courbes (distinctes ou non) dégénérées en n droites concourantes. Si les points de concours sont distincts et non alignés, on obtient un faisceau quelconque du réseau $ax^n+by^n+cz^n=0$ dont la hessienne est formée du triangle de référence avec la multiplicité $n-2$. Si les points de concours sont distincts mais alignés, on obtient seulement le cas banal d'une droite fixe associée à une droite variable dans un faisceau, dont la hessienne est indéterminée. Si deux des points de concours sont confondus, on obtient soit un faisceau de C^n avec n points bases inflexionnels maximaux alignés, à tangentes concourantes, soit un faisceau de C^n ayant un point multiple d'ordre $n-1$ à tangente unique fixe, et n points bases alignés. Si les trois points de concours sont confondus on obtient deux éventualités: les faisceaux de C^n ayant un point multiple d'ordre $n-1$ à tangente unique fixe, ayant tous les points bases, sauf un au plus, confondus avec ce point multiple (cas particulier du précédent); lorsque $n\geqq 9$, on peut avoir des faisceaux de C^n ayant un point multiple d'ordre $n-1$ avec deux tangentes fixes de multiplicités complémentaires. Dans ce dernier cas seulement, la hessienne, toujours décomposée, peut comporter une ou plusieurs composantes non rectilignes. *L. Gauthier* (Nancy).

Chow, Wei-Liang. On compact complex analytic varieties. Amer. J. Math. **71**, 893–914 (1949).

Soit P_n l'espace projectif complexe à n dimensions (complexes). L'auteur démontre la conjecture classique: toute variété analytique complexe, partout régulière dans P_n, est une variété algébrique. La démonstration comporte deux parties: topologique et algébrique. Dans la partie topologique, le "degré" d'une variété analytique irréductible V, de dimension r, est défini comme l'entier d tel que le $(2r)$-cycle porté par V soit homologue à d fois le $(2r)$-cycle porté par un sous-espace projectif de dimension (complexe) r. Cette définition est licite parce qu'on peut enlever de V un sous-espace fermé de dimension réelle $\leqq 2r-2$, de manière que l'ensemble restant soit une véritable variété (au sens topologique), connexe, de dimension (réelle) $2r$. La théorie topologique des intersections peut être appliquée: le fait que, en un point commun isolé, la multiplicité d'intersection de deux variétés analytiques complexes (de dimensions complémentaires) est toujours positive, entraîne ceci: si deux variétés analytiques V et W, de degrés p et q, et de dimensions complémentaires, ont en commun un point isolé, sa multiplicité d'intersection est $\leqq pq$. De là résulte que V est algébrique, en vertu du théorème suivant, qui constitue la deuxième partie (algébrique) de la démonstration: soit V un élément de variété analytique (de dimension complexe r) en un point a; si V n'est pas algébrique, alors, pour tout entier p, existe une variété algébrique W (de dimension $n-r$) passant par a, telle que la multiplicité d'intersection de V et W au point a soit plus grande que p fois le degré de W. L'auteur montre en outre que cette caractérisation (locale) des variétés algébriques peut être considérée comme une extension du critère classique qui caractérise les nombres algébriques par la considération d'inégalités diophantiennes. Pour finir, l'auteur montre que toute transformation méromorphe qui est définie sur une variété analytique de P_n et l'applique dans P_m est une transformation rationnelle.
H. Cartan (Paris).

Northcott, D. G. An inequality in the theory of arithmetic on algebraic varieties. Proc. Cambridge Philos. Soc. **45**, 502–509 (1949).

A point P having in n-dimensional projective space S_n homogeneous coordinates $(\xi)=(\xi_0, \cdots, \xi_n)$ is called algebraic if all the ratios ξ_i/ξ_j are algebraic numbers. Then in any given number field K containing all these ratios the ξ_i generate a fractional ideal Ω, belonging to the ring of algebraic integers in K. Moreover, denoting by R the field of rational numbers, $N_{K/R}\Omega$ is a fractional ideal in R, admitting a unique positive generator, $|N_{K/R}\Omega|$ say. Consider the positive number

$$C(P)=\{\prod_\sigma(|\xi_0^\sigma|+\cdots+|\xi_n^\sigma|)/|N_{K/R}\Omega|\}^{1/(K:R)},$$

where the product is to be extended over all the relative isomorphisms σ of K over R into the complex field, and $(\xi_0^\sigma, \cdots, \xi_n^\sigma)$ is the transform of P by σ. It is easily seen that $C(P)$, called the arithmetical complexity of P, depends only on P and not on K. The purpose of this paper is to consider how $C(P)$ varies when P undergoes a mapping T of a certain type. The main result is expressed by the inequality $A\leqq C^m(P)/C(P^T)\leqq B$, concerning an algebraic point P varying on an indivisible (i.e., absolutely irreducible) algebraic variety V embedded in S_n, free from multiple points and defined over an arbitrary number field k. The mapping T associates at P the point P^T of coordinates $(T_0(\xi), \cdots, T_n(\xi))$, where $T_0, \cdots, T_n$ are $n+1$ forms on V of the same degree m, defined on k and having no common zero on V. In the formula above A and B denote positive constants depending only on V and T.

The inequality is established by using A. Weil's "theory of distributions" [Acta Math. **52**, 281–315 (1929)], described here in detail with the addition of some new results [cf. especially lemma 3]. *B. Segre* (Bologna).

Northcott, D. G. A further inequality in the theory of arithmetic on algebraic varieties. Proc. Cambridge Philos. Soc. **45**, 510–518 (1949).

This paper is a sequel to the previous one, and we refer to the preceding review for the notations. It proves the inequality $C^m(P)\leqq MC(P^T)$, where now T denotes a more general mapping of the variety V, called a regular mapping, m is an integer attached to it, M is a positive constant depending only on V and T, and the inequality holds for every algebraic point P of V, with only possible exceptions for the points of certain subvarieties defined by T. By introducing on an elliptic curve the Weierstrassian uniformizing parameter u, one can define on it certain mappings of the types considered in this paper and in the previous one, i.e., $u\rightarrow Nu$, $u\rightarrow u-\alpha$, $u\rightarrow 2u-\alpha$, where N is a positive integer and α corresponds to an algebraic point. The general results previously obtained give then some new arithmetical properties of the elliptic cubics. *B. Segre* (Bologna).

Northcott, D. G. The values taken by a rational function on an algebraic variety. Proc. Cambridge Philos. Soc. **45**, 675–677 (1949).

Certain rather obvious connections between divisor and specialization theory, used in the previous two papers, are established here explicitly. In particular, it is proved that a zero or infinity in the sense of divisor theory is still a zero or infinity in the sense of specialization theory, and vice versa. *B. Segre* (Bologna).

Derwidué, L. Résolution des singularités d'une surface algébrique au moyen de transformations crémoniennes. Bull. Soc. Roy. Sci. Liége **16**, 275–289 (1947).

The resolution, by means of Cremona transformations in S_3, is to ordinary singularities consisting of multiple curves with distinct tangent planes in general and with pinch points at which two tangent planes coincide, the multiple curve itself having no singularities except double points with distinct tangents which are not polar singularities. Let F be an algebraic surface in S_3, with no reducible components. A point of F which is a base point of the variable part of the system of first polar curves on F is called a polar singularity of F. The first step in the resolution is the elimination of polar singularities. This is done by applying Cremona transformations which have an isolated fundamental point at a polar singularity. Since the polar singularities, distinct or neighboring, are among the intersections of distinct polar curves, there is a finite number of them, and so they can be eliminated by a finite number of such transformations. The particular transformations used are shown to introduce no new polar singularities but pinch points. The same type of transformation can then be used to simplify the singular curves of the surface. There is obtained a surface F' on which the singular curve has itself no singularities but ordinary double points.

A general plane section of F' is a curve having singular points on the singular curves of F'. In this plane a Cremona transformation is set up having ordinary fundamental points at these singularities. By allowing the plane to vary in a pencil a Cremona transformation in S_3 is obtained. Operating on F' with this transformation is shown to simplify the general point of the singular curve in the usual manner and to introduce only ordinary additional singularities. The desired result then follows. The method of proof is entirely geometric, and is not always convincing. *R. J. Walker* (Ithaca, N. Y.).

Enriques, F. Sur la démonstration géométrique d'un théorème de Picard, concernant les surfaces algébriques. Revista Acad. Ci. Madrid **43**, 75–77 (1949).

Severi, in the course of his attempted geometrical proof and extension [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **17**$_1$, 465–470 (1908)] of Picard's theorem (that the adjoints to the plane sections of an algebraic surface cut on the general plane section its complete canonical series) purported to prove the following lemma. Given on an algebraic surface an irreducible linear system $|D|$, effectively free from base points, and an irreducible curve C, which defines a continuous system at least ∞^1, virtually free from base points; if the system $|E|=|D-C|$ exists, it cuts a complete series on the general curve of $|D|$. In the present note it is proved, on the contrary, that given on the surface any irreducible curve C, and an irreducible linear system $|L|$, of freedom at least 2, without fundamental curves, then $|D|=|sL|$, for sufficiently high values of s, is such that $|E|=|D-C|$ exists, but cuts an incomplete series on the general curve of $|D|$. It is pointed out that Severi's error lies in the misapplication of his own criterion of equivalence. *P. Du Val* (Athens, Ga.).

Jongmans, F. Étude sur le genre linéaire des surfaces algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 470–483 (1949).

The author gives improvements on existing inequalities connecting the linear genus of an algebraic surface with the

This review is identical with the one in V.10, p. 473. In regard to the error attributed to Severi in this rev. see the correction above to the earlier rev.

arithmetic and geometric genera, valid for fairly extensive classes of surfaces. *J. A. Todd* (Cambridge, England).

Nollet, Louis. Les systèmes linéaires de courbes algébriques planes et leurs classifications. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 730–742 (1949).

Expository lecture. Cf. Mém. Soc. Roy. Sci. Liége (4) 7, 469–554 (1947); these Rev. 9, 461. *D. Pedoe* (London).

Nollet, Louis. Contribution à l'étude des surfaces algébriques dont le système canonique est composé au moyen d'une involution. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 886–894 (1949).

Classification of surfaces whose birational invariants satisfy certain inequalities. *D. Pedoe* (London).

Muhly, H. T. The irregularity of an algebraic surface and a theorem on regular surfaces. Bull. Amer. Math. Soc. 55, 940–947 (1949).

Let Σ be a field of algebraic functions of two variables over a field k of characteristic 0 which is algebraically closed in Σ. If U is a normal model of Σ, the sections of U by hypersurfaces of degree m of the ambient space determine a complete linear system on U. It is proved that the deficiency of this system does not depend on m when m is large enough; this ultimate deficiency is a relative invariant of U, which is denoted by $\delta(U)$. If U has no singularity, then it is well known that $\delta(U)$ is the irregularity q of Σ. It is proved that $\delta(U)\leqq q$ for any normal model U, and that $\delta(U)\leqq\delta(U')$ if U and U' are normal models such that the birational correspondence between them has no fundamental point on U'. This yields a new proof of the absolute invariance of the irregularity.

A normal model U for which $\delta(U)=0$ is called regular; the question of the existence of such models when $q>0$ remains open. It is proved that, if U is regular, then there are arithmetically normal derived models U' of U with the property that every nonsingular hyperplane section of U' is arithmetically normal. Assuming that Σ is regular, let $\{\xi_1, \xi_2\}$ be a transcendence base of Σ/k. Then it is proved that there exist integers h such that the integral closure of $R_h=k[\xi_1^h, \xi_2^h]$ in Σ has a (linearly independent) base over R_h. *C. Chevalley* (New York, N. Y.).

Conforto, Fabio. Sopra le trasformazioni in sè della varietà di Jacobi relativa ad una curva di genere effettivo diverso dal genere virtuale, in ispecie nel caso di genere effettivo nullo. Ann. Mat. Pura Appl. (4) 27, 273–291 (1948).

L'autore considera la varietà, quasi abeliana di Jacobi V_π, relativa ad una curva di genere virtuale π, ottenuta da una curva di genere effettivo p considerando su questa d_1 coppie neutre a punti distinti e d_2 coppie neutre a punti coincidenti, in modo che sia $\pi=p+d_1+d_2$, e precisa il problema di determinare le trasformazioni in sè di questa varietà V_π. Mostra che il problema ha un aspetto aritmetico che si collega a relazioni che generalizzano quelle note di Hurwitz. Nel caso $p=0$, l'autore trova tutte le trasformazioni in sè della varietà V_π, mostrando che esse costituiscono un gruppo T formato da infinite schiere ∞^π (tranne il caso $\pi=d_1=1$, $d_2=0$), tra le quali sono contenute in particolare le due schiere delle trasformazioni di prima e seconda specie. Le trasformazioni sono tutte birazionali se $d_1=0$, $d_2=0$ (caso abeliano). Se $d_1=0$ ($\pi=d_2$), il gruppo T è un gruppo continuo a $\pi(\pi+1)$ parametri, isomorfo al gruppo delle affinita (non degeneri) di uno spazio lineare S_π. Se invece $d_2=0$ ($\pi=d_1$) il gruppo T è isomorfo al gruppo delle trasformazioni cremoniane monoidali (non degeneri) di uno spazio lineare S_π. Nel caso $d_1\neq0$, $d_2\neq0$, il gruppo T è un gruppo continuo misto ad infinite schiere costituito da trasformazioni, generalmente, trascendenti. *M. Piazzolla-Beloch.*

Brenci, Maria Teresa. Sulla varietà quasi abeliana di Jacobi di genere effettivo nullo e genere virtuale due. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 7, 458–483 (1948).

Let V_π be the Jacobi variety of a curve with virtual genus π, which may be obtained by fixing π neutral pairs with distinct points on a curve with effective genus equal to zero. According to a result of the reviewer [see the preceding review], V_π possesses infinitely many systems ∞^π of birational transformations in itself, which are represented by linear congruences among the integrals, to be virtually considered of the first kind on V_π. Since F. Severi [Funzioni quasi abeliane, Pontificae Academie Scientiarum Scripta Varia, v. 4, Vatican City, 1947; these Rev. 9, 578] has shown that, in the case referred to, one may take, as a model for V_π, a projective linear space S_π, one may deal with a certain group, composed by infinitely many systems ∞^π of Cremona transformations in an S_π. The author makes such a study for $\pi=2$, in which case the Jacobi variety reduces to a plane. By a suitable choice of the integrals and of the coordinates in the plane, one is led to examine the group of the Cremona transformations of the type:

$$(1)\qquad x_1'=b_1x_1^{a_{11}}x_2^{a_{12}},\quad x_2'=b_2x_1^{a_{21}}x_2^{a_{22}},$$

where b_1 and b_2 are arbitrary numbers different from zero, and a_{11}, a_{12}, a_{21}, a_{22} are arbitrary integers, which are subject to the condition (2) $a_{11}a_{22}-a_{12}a_{21}=\pm1$. If the integers a_{11}, a_{12}, a_{21}, a_{22} are fixed in any way so that (2) is satisfied, one obtains a system ∞^2 of the group, by letting b_1, b_2 vary in (1). The author determines in any possible case the order and the basis points of the homaloidal net attached to the transformation (1); the study of the basis points requires a thorough examination of multiple points which are infinitely near, upon linear or superlinear branches; the configuration of these multiple points depends upon arithmetic properties of the a_{ik}, expressed, e.g., in terms of the expansion in a continued fraction of a_{22}/a_{21}. *F. Conforto* (Rome).

Predonzan, Arno. Intorno agli S_k giacenti sulla varietà intersezione completa di più forme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 238–242 (1948).

Nella presente nota, l'autore estende un risultato di U. Morin e B. Segre sulla condizione necessaria e sufficiente affinchè esistano spazi lineari S_i sulla forma generale di ordine n dell' S_r, al caso più generale della varietà intersezione di più forme. *M. Piazzolla-Beloch* (Ferrara).

Predonzan, Arno. Sull'unirazionalità della varietà intersezione completa di più forme. Rend. Sem. Mat. Univ. Padova 18, 163–176 (1949).

U. Morin [Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 298–302 (1942); ces Rev. 8, 527] a établi que l'hypersurface de degré n d'un espace S_r (r assez grand) est unirationnelle, c'est-à-dire que ses coordonnées peuvent s'exprimer en fonctions rationnelles de $r-1$ paramétres non homogènes, le même point s'obtenant un nombre fini de fois. L'auteur étend cette propriété aux variétés V_s à s dimensions, intersections de $r-s$ formes de l'espace S_r où r est assez grand. La limitation de r à partir d'un nombre r_0 défini par une loi de récurrence faisant intervenir le nombre

a_i ($i=1, 2, \cdots, \nu$) des formes de degré i montre que d'après un résultat précédent de l'auteur [voir l'analyse ci-dessus] les V considérées contiennent des espaces linéaires S de dimension $r_{\nu-1}$. Partant du cas $\nu=1$ il prouve par récurrence que l'on peut exprimer rationnellement les coordonnées de V à partir de s paramètres, les coefficients des équations paramétriques dépendant rationnellement de ceux des formes génératrices de V et des paramètres qui définissent un espace linéaire contenu dans V, d'où l'unirationalité annoncée.

B. d'Orgeval (Grenoble).

✶Hohn, Franz Edward. Curves on Cayley's Dianodal Surface. Abstract of a Thesis, University of Illinois, 1940. ii+10 pp.

The surface in question is the locus of eighth nodes of quartic surfaces having nodes at seven fixed points in S_3. It has a parametrisation in terms of ϑ functions of three variables, subject to one relation $\vartheta(u_1, u_2, u_3)=0$. This is used throughout the paper. The author considers the set of 29 curves consisting of a general plane section, the lines joining the seven points by pairs, and the rational cubics through six of them. It is not proved that these are a base for algebraic curves on the surface, though this is surmised; their intersection matrix is found and (from intersection numbers) the linear expressions of the following curves in terms of these are obtained: the neighborhoods of the seven base points (which are elliptic triple points of the surface); the 21 elliptic quartics, intersections of two quadric cones passing through five of the seven points and having their vertices at the other two; the 35 elliptic cubics, intersections of the plane joining three of the seven points with the cubic surface passing through these three and having nodes at the other four; the sextic of genus three, locus of vertices of quadric cones through the seven points; the dianodal curves of order 18, locus of ninth nodes of quartic surfaces having nodes at the seven points and a general point of the surface. All curves given by the vanishing of ϑ functions of order 2. The linear systems of surfaces tracing these are also investigated.

P. Du Val (Athens, Ga.).

Godeaux, Lucien. Les points unis des involutions cycliques appartenant à une surface algébrique. Ann. Sci. École Norm. Sup. (3) **65**, 189–210 (1948).

L'auteur considère une surface algébrique contenant une involution I_p cyclique d'ordre premier impaire p, n'ayant qu'un nombre fini de points unis et en construit comme modèle projectif une surface normale F d'un espace linéaire S_r, sur laquelle I_p est déterminée par une homographie cyclique T de l'espace ambient S_r, possédant p axes ponctuels $\sigma_0, \cdots, \sigma_{p-1}$. Il désigne par $|C_i|$ le système appartenant à l'involution, des courbes découpés sur F par les hyperplans de S_r passant par $\sigma_0, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{p-1}$ ($i=0, 1, 2, \cdots, p-1$). Le systeme $|C_0|$ est dépourvu de points base, les autres ont pour points base les points unis de I_p. En rapportant projectivement les courbes de $|C_0|$ aux hyperplans d'un espace linéaire de dimension convenable, on obtient une surface normale Φ image de l'involution. À un point uni A de la surface F correspond sur Φ un point de diramation A'. Lorsque le point A est uni "parfait" les courbes C_0 passant par A acquièrent en ce point la multiplicité p, les tangentes étant variables. Le point de diramation correspondant A' est multiple d'ordre p pour la surface Φ, le cône tangent étant rationnel et irréductible. Lorsque le point A est uni "non parfait" pour l'involution, simple pour la surface F, il exist dans le domaine du premier ordre de ce point deux points unis pour l'involution. Soient a_1, a_2 les tangentes à F en A passant par ces points. Les courves C_0 passant par A, acquièrent en ce point une multiplicité $s_1<p$, les tangentes étant confondues avec a_1, a_2. Appellons $|C_0'|$ le système formé par ces courbes. Les courbes C_0' assujetties à toucher en A une droite distincte de a_1, a_2, forment un système $|C_0''|$ dont les courbes ont en A une multiplicité $s_2<p$. Et ainsi de suite ($0\leqq s_1<s_2<\cdots<p$). On construit ainsi une suite de systèmes linéaires $|C_0'|, |C_0''|, \cdots$, dont les courbes ont en A des multiplicités croissantes, les tangentes en ce point étant confondues avec deux droites sauf pour le dernier système, dont les courbes ont en A le multiplicité p et des tangentes variables. Au point A est attaché un entier positif R ($1<k<p$), tel que, si les courbes C_0^i ont λ_i tangentes en A confondues avec a_2 et μ_i tangentes confondues avec a_1, on ait: $\lambda_i+k\mu_i\equiv 0 \pmod p$, où $\lambda_i+\mu_i<k$. Parmi les systemes $|C_1|, |C_2|, \cdots, |C_{p-1}|$, il en est deux pour lesquels A est un point base simple. Les courbes de l'un, $|C_1|$, de ces systèmes touchent la droite a_2 en A, et les courbes de l'autre, $|C_k|$, touchent la droite a_1 en ce point. Les courbes C_1, C_k rencontrent les courbes $C_0', C_0'', \cdots$, en p point confondus en A. Déterminer la structure du point A revient à déterminer les différentes branches d'origine A appartenant aux courbes $C_0', C_0'', \cdots$, et leur comportement en A. On en deduit alors la structure du point de diramation A' de Φ, c'est-à-dire sa multiplicité et la configuration formée par les cônes tangents à la surface Φ en ce point. L'auteur donne ensuite une application de la méthode au cas particulier où le nombre premier p est de la forme $p=10\eta+3$, et $k=5$.

M. Piazzolla-Beloch.

Godeaux, Lucien. Sur les points unis des involutions cycliques appartenant à une variété algébrique à trois dimensions. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 695–700 (1948).

[Voir même Bull. Cl. Sci. (5) **34**, 419–425, 518–530 (1948); ces Rev. **10**, 474, 475.] Dans cette note l'auteur démontre que: si une variété algébrique à trois dimensions contient une involution cyclique d'ordre premier $p=2\nu+1$, possédant un point uni de seconde espèce auquel est infiniment voisin un unique point uni parfait, on peut prendre comme image de l'involution une variété normale pour laquelle le point de diramation correspondant est multiple d'ordre $\nu^2+\nu+1$, le cône tangent se composant d'un cône d'ordre ν^2 et d'un cône d'ordre $\nu+1$ se rencontrant suivant un cône d'ordre ν. Le cône d'ordre ν^2 projette une surface de Veronese généralisée représentant les courbes d'ordre ν d'un plan et le cône d'ordre $\nu+1$ projette une surface réglée.

M. Piazzolla-Beloch (Ferrara).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. IV. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 532–541 (1949).

Dans une des notes précédentes avec le même titre [même Bull. Cl. Sci. (5) **35**, 285–292 (1949); ces Rev. **11**, 205] l'auteur a considéré un cas où la surface Φ qui représente une involution cyclique d'ordre premier p appartenant à une surface algébrique possède un point de diramation isolé dont le cône tangent se décompose en trois parties, et précisément deux plans et un cône de quatrième ordre. Ici, en généralisant la question, il suppose que le cône tangent à la surface Φ au point de diramation en question se décompose en deux plans σ_1, τ et un cône σ_n d'ordre $\mu_1>1$. En supposant encore que τ rencontre σ_1 et σ_n chacun suivant une droite tandis que σ_1 et σ_n ne se rencontrent pas, et que Φ possède un point double conique infiniment voisin du point de diramation sur la droite $\tau\sigma_n$, alors Φ possède un autre

point double conique infiniment voisin du point de diramation sur la droite $\sigma_1\tau$, ou bien on a: $p=(3\xi+2)(3\eta\pm1)+3$; $\mu_1=3\eta\pm1$; $\alpha=3\xi+2$; ξ, η étant des entiers positifs. Enfin un exemple de ce dernier cas, avec $p=19$, $\alpha=8$, $\mu_1=2$.

E. G. Togliatti (Gênes).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. V. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 636–641 (1949).

Voici encore une continuation d'une recherche du même auteur sur la structure des points de diramation isolés d'une surface algébrique multiple [voir l'analyse ci-dessus]. En conservant les notations des quatre notes précédentes, l'auteur considère ici un cas où $p=41$, $\alpha=25$. Dans ce cas le point de diramation isolé A' de la surface multiple Φ a pour Φ la multiplicité 6; et son cône tangent se décompose en quatre parties, c'est-à-dire trois plans et un cône cubique; la singularité de Φ au point A' est équivalente à l'ensemble de quatre courbes rationnelles, dont chacune rencontre seulement la précédente et la suivante en un point. Dans la note présente l'auteur abandonne ses locutions de "points unis parfaits ou non parfaits" d'une involution sur une surface algébrique pour adopter celle de "points unis de 1[re] ou de 2[me] espèce" introduite par l'école italienne.

E. G. Togliatti (Gênes).

Godeaux, Lucien. Sur la construction de surfaces non rationnelles de genres zéro. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 688–693 (1949).

Metelka, Josef. Trois chapitres sur les transformations monoïdales. Rozpravy II. Třídy České Akad. **58**, no. 10, 57 pp. (1948). (Czech. French summary)

L'auteur appelle monoïdale une transformation birationnelle dans laquelle les variétés V^n_{r-1} transformées des hyperplans de S_r ont un S_{r-2} fixe, multiple d'ordre $n-1$. Les autres variétés fondamentales sont des S_k supposés non infiniment voisins entre eux, ni infiniment voisins à S_{r-2}. Leurs nombres a_k sont liés par: $\sum(k+1)a_k=r(n-1)$. La variété V_k transformée d'un S_k arbitraire est d'ordre:

$$n_k=(n-1)k+1-\sum(k+i+1-r)a_i.$$

On peut aussi définir une telle transformation monoïdale comme transformant par homographie les points d'un hyperplan qui varie dans un faisceau. La transformation inverse est également monoïdale. Dans la seconde partie du mémoire, l'auteur étudie le produit de deux transformations monoïdales ayant le même S_{r-2} singulier. Il résulte de cette étude que toute telle transformation monoïdale est le produit d'un nombre fini de transformations monoïdales quadratiques. Dans la troisième partie du mémoire, l'auteur étudie les transformations monoïdales cycliques, notamment de période paire, en liaison avec les propriétés des homographies cycliques et des courbes rationnelles normales.

L. Gauthier (Nancy).

Differential Geometry

Gougenheim, André. Sur une nouvelle famille de planisphères conformes permettant de représenter la terre à l'intérieur d'un contour fermé quasi elliptique. C. R. Acad. Sci. Paris **230**, 369–371 (1950).

A conformal projection of the terrestrial sphere within an elliptical boundary is given by

$$\tan\left(\tfrac{1}{2}Z\tan\tfrac{1}{2}\alpha\right)=\tan\tfrac{1}{2}\alpha\tanh\tfrac{1}{2}\zeta,$$

where Z is the complex variable for the projection coordinates, ζ the complex variable for the longitude and isometric latitude on the sphere, and α a projection parameter.

N. A. Hall (Minneapolis, Minn.).

Mirguet, Jean. Sur une généralisation des orthosurfaces. C. R. Acad. Sci. Paris **230**, 48–50 (1950).

The author considers sets which satisfy four rather complicated conditions. These sets are more general than unions of orthosurfaces of Bouligand [Introduction à la géométrie infinitésimale directe, Vuibert, Paris, 1932, p. 78] and have the property that at every point of one of them its contingent reduces to two planes or to two nappes of a convex cone.

H. L. Smith (Baton Rouge, La.).

Mitrinovitch, Dragoslav S. Problème, dont la solution dépend d'une équation de Riccati, relatif aux asymptotiques d'une surface réglée. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 89–92 (1939).

Cette note est relative aux surfaces réglées

$$[x=x_0(u)+a(u)\cdot v,\quad y=y_0(u)+b(u)\cdot v,\quad z=z_0(u)+c(u)\cdot v]$$

dont les asymptotiques sont définiés par une équation différentielle $dv/du+f(u)v^2+\varphi(u)+\psi(u)=0$ donnée d'avance. Si l'on se donne arbitrairement a, b, c, la détermination des surfaces en question revient à celle des trois fonctions x_0, y_0, z_0 de u, et l'auteur montre que cette détermination se ramène à l'intégration d'une équation de Riccati suivie de quadratures.

P. Vincensini (Marseille).

Viguier, Gabriel. Les développantes généralisées du second ordre d'une courbe plane. C. R. Acad. Sci. Paris **229**, 462–464 (1949).

Verf. betrachtet eine ebene Kurve $\mathfrak{x}(t)$ und ordnet dieser eine zweite Kurve $\mathfrak{l}(t)$ zu. Die Zuordnung erfolgt in zwei Schritten: zunächst wird im Punkt M der Kurve $\mathfrak{x}(t)$ das Segment $MN=\tau(t)$ aufgetragen. Für alle Werte t des Kurvenparameters entsteht so die verallgemeinerte Envolvente $\mathfrak{e}(t)$. Im Punkt N dieser Envolvente wird nunmehr längs der Tangente wiederum ein Segment $NP=\chi(t)$ abgetragen. Die so entstehende Kurve $\mathfrak{y}(t)$ heisst verallgemeinerte Envolvente zweiter Ordnung der Basiskurve $\mathfrak{x}(t)$. Ihre Tangentenvektoren bilden die Ortsvektoren der adjungierten Kurve $\mathfrak{l}(t)$. Die analytische Formulierung dieser Konstruktionsvorschrift führt auf interessante Sonderfälle, je nachdem die entstehenden Differentialgleichungen für $\tau(t)$ bzw. $\chi(t)$ vom Riccatischen oder vom Abelschen Typus sind.

M. Pinl (Dacca).

Fáry, István. Sur la courbure totale d'une courbe gauche faisant un nœud. Bull. Soc. Math. France **77**, 128–138 (1949).

The author proves that the total curvature of a closed curve in the ordinary space is equal to the mean value of the total curvatures of its orthogonal projections (the total curvature of a plane curve being defined as the integral of the absolute value of the curvature). From this the known fact that the total curvature of a space curve is at least 2π follows immediately. By means of the above lemma and known properties of the projections of knots it is further proved that the total curvature of a knotted curve is at least 4π. Thus a question raised by K. Borsuk [Ann. Soc. Polon. Math. **20** (1947), 251–265 (1948); these Rev. **10**, 60] is answered. [The reviewer remarks that the author's formulation of the definition of a knot is incorrect, but this does not affect the results.]

W. Fenchel.

Rozet, O., et Legrain-Pissard, N. Sur les surfaces à lignes de courbure planes dans les deux systèmes. Bull. Soc. Roy. Sci. Liége **18**, 259–265 (1949).

The purpose of this note is to add to the results of a paper by the same authors [same Bull. **13**, 10–15 (1944); these Rev. **7**, 75], on surfaces with plane lines of curvature. Two of the results may be stated as follows. If a surface has plane lines of curvature, so does any surface parallel to it. If, moreover, the surface is a surface of Weingarten, then the sum of its principal radii of normal curvatures is a constant. *V. G. Grove* (East Lansing, Mich.).

Rozet, O. Sur les formes différentielles des surfaces et plus généralement, des variétés. Bull. Soc. Math. Belgique **1** (1947–1948), 1–4 (1949).

Report of a lecture.

Backes, F. Sur les familles de surfaces dont les lignes de courbure ont même projection cylindrique. Bull. Soc. Math. Belgique **1** (1947–1948), 20–23 (1949).

Marcus, F. Un teorema di geometria differenziale. Boll. Un. Mat. Ital. (3) **4**, 109–111 (1949).

The purpose of this note is to prove the following theorem. If the lines of curvature of two surfaces S_1, S_2 in one-to-one correspondence correspond, and if the lines joining corresponding points are normal to S_1, they are also normal to S_2, excluding the case of molding surfaces. *V. G. Grove.*

Lorent, H. Une famille de transformations des courbes et des surfaces. Bull. Soc. Roy. Sci. Liége **18**, 300–311 (1949).

A study is made of a transformation of curves defined as follows: let a point $A(x', y')$ be referred to a rectangular coordinate system; let a point $B(x, y)$ be chosen on the bisector of the angle AOX; suppose $x=x(x', y')$, $y=y(x', y')$. The locus $\bar{C}$ of B (or A) as A (or B) describes a curve C is the transform of C to be studied. This transformation is generalized to surfaces as follows. Let $A(x', y', z')$ be referred to a rectangular coordinate system. The point $B(x, y, z)$ is taken on the bisector of the angle which OA makes with the (x, y)-plane; and $x=x(x', y', z')$, $y=y(x', y', z')$, $z=z(x', y', z')$. The locus $\bar{S}$ of B (or A) as A (or B) generates a surface S is the transform of S to be studied. Several special cases are described. *V. G. Grove* (East Lansing, Mich.).

Dekker, David B. Hypergeodesic curvature and torsion. Bull. Amer. Math. Soc. **55**, 1151–1168 (1949).

A hypergeodesic curve on a surface satisfies a second order differential equation similar in structure to that for geodesic curves except that the coefficients are arbitrary point functions instead of the Christoffel symbols. If the osculating planes of a hypergeodesic family at each point of the surface form an axial pencil, then the family is a family of union curves. Union curvature and union torsion relative to such a family have been defined earlier as generalizations of geodesic curvature and geodesic torsion. In the present paper the author gives a definition of hypergeodesic curvature and hypergeodesic torsion relative to a family of hypergeodesic curves which generalizes the definitions of union curvature and union torsion. Also, a geometric condition that a hypergeodesic be a plane curve is found.

A. Fialkow (Brooklyn, N. Y.)

Wilkins, J. Ernest, Jr. Some remarks on ruled surfaces. Bull. Amer. Math. Soc. **55**, 1169–1176 (1949).

With the aid of his power series expansion for the equation of a nondevelopable ruled surface S in ordinary projective space [Amer. J. Math. **67**, 71–82 (1945); these Rev. **6**, 216], the author obtains additional results. He finds the double points of cubic surfaces of a one-parameter family of such surfaces [equation (2)] having fifth order contact with S at a point x. There is only one double point of a cubic surface (2) in the tangent plane to S at x, and this point is a double point for all cubic surfaces of the family. Except when S is a special surface characterized by the vanishing of certain invariants, there are two cubic surfaces (2) each of which has a (unique) double point which is not in the tangent plane to S at x. Double points are also found in the cases in which S is characterized by the vanishing of one or more of the quantities $I^3-\gamma J$, $\gamma P^2-2IP+J$, $\gamma_u\gamma_v-\gamma\gamma_{uv}$, D. If none of these four quantities vanishes, there are three cubic surfaces (2) which have a double point not in the tangent plane. If exactly one of these quantities vanishes, there are two such cubic surfaces. If exactly two vanish there is one such cubic surface. If exactly three vanish, there are two such cubic surfaces, one of which is S itself and is not Cayley's cubic scroll. If all four vanish, there is one such cubic surface, namely S itself, which is Cayley's cubic scroll. The curve of intersection of S with the tangent plane to S at x is studied. There exists a one-parameter family of noncomposite cubic curves with a node at the origin and asymptotic tangents for nodal tangents, each of which has fifth order contact with the curve. If it is not insisted that the nodal tangents be the asymptotic tangents, the one-parameter family can have sixth order contact with the curve. The line of inflexions of a general member of this family depends upon a parameter E. As E varies this line of inflexion envelops a conic which turns out to be the osculating conic of the curve of intersection of S with the tangent plane to S at x. *P. O. Bell.*

∗Stewart, James Collier. Geodesic Correspondences Between Surfaces of Revolution. Abstract of a Thesis, University of Illinois, 1946. ii+7 pp.

In this abstract the author mentions briefly and without proofs some results obtained by considering the geodesic mapping of surfaces (H-surfaces) whose curves $K=$constant ($K=$Gaussian curvature) are geodesic parallels. Typical sample of theorems: an H-surface that is not a surface of revolution can be in geodesic correspondence only with another H-surface and the two surfaces must allow linear elements whose ratio is a constant. *V. Hlavatý.*

Löbell, Frank. Ein differentialgeometrischer Operator in der Theorie der Flächenabbildungen. Arch. Math. **2**, 17–23 (1949).

Let $x=x(u, v)$ be the vector equation of a surface in Euclidean 3-space expressed in regular parameters, and let $w(u, v)=|x_u \times x_v|$. The invariant differential operator $\mathfrak{D}=(x_v\partial/\partial u-x_u\partial/\partial v)/w$ is introduced, and its utility for expressing relations in the theory of surfaces and surface mappings is explored. *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Weingartens charakteristische Gleichung und eine ähnliche Differentialgleichung in der Theorie der Flächenabbildungen. Arch. Math. **2**, 96–102 (1950).

The author shows that certain apparently unrelated problems in deformation of surfaces, statics of membranes, and mappings of pairs of surfaces are actually equivalent. They

each reduce to finding surfaces $y=y(u,v)$ corresponding to a given surface $x=x(u,v)$ in such a way that the spread vector [Spreizvector] of the mapping is a given function of position on x. The solution of these problems can be made to depend on the solution of a second-order partial differential equation corresponding to Weingarten's characteristic equation. *S. B. Jackson* (College Park, Md.).

Baier, Othmar. Eine geometrische Ableitung des Gauss-Bonnetschen Integralsatzes. Arch. Math. **2**, 105–109 (1950).

A proof is given for the classical Gauss-Bonnet theorem for a two-dimensional curved region. The method involves an approximation to the boundary of the region by a rectilinear polygon lying in the enveloping space. From a study of this approximation and a limiting process the usual formula is derived. *C. B. Allendoerfer.*

Grove, V. G. On the R_λ-associate of a line. Proc. Amer. Math. Soc. **1**, 20–22 (1950).

The notion of Popa's correspondence [Acad. Roum. Bull. Sect. Sci. **23**, 31–33 (1942); these Rev. **9**, 532] is extended in this note to the asymptotic net on a surface. This extension yields new geometric descriptions of the R_λ-associate of a line and the R_λ-derived curves (objects introduced by the reviewer [Trans. Amer. Math. Soc. **46**, 389–409 (1939); these Rev. **1**, 85]). Let x denote a generic point of a surface S referred to asymptotic parameters u, v. Let h_1 denote a line which passes through x, but does not lie in the tangent plane to S at x, and let h_2 denote the reciprocal of h_1 with respect to a quadric of Darboux. Let C denote a curve through x defined by the differential equation $dv-\lambda du=0$, let l denote the tangent to C at x, and let y denote a point of C near to x given by $y=x(u+\Delta u, v+\Delta v)$. The tangent to the asymptotic curve $v=\text{constant}$ at y intersects the plane determined by h_1 and the tangent to the $v=\text{constant}$ curve at x in a point, the limit of whose projection on the tangent plane to S at x from a point z of h_1 as y approaches x along C is a point ρ which is independent of the choice of z on h_1. A point σ is similarly defined on the asymptotic v-tangent to S at x. The line h determined by the points ρ, σ is the R_μ-associate of h_2 $(\mu=-\lambda)$. The line h envelops a conic as λ varies. Let T denote the point of contact of h with the conic, and let m denote the line joining x and T. The conjugate of m is the R_λ-correspondent of l. The line m intersects the conic in the points T and T'; the tangent to the conic at T' is the R_λ-associate of h_2. The locus of the intersection of l and h as l varies in a pencil on x is a rational cubic curve. [This cubic is a special case of the cubic (8.7) derived similarly by the reviewer in the paper cited above.] If the line h_1 is the R-conjugate line, this cubic is the Hessian of the characteristic cubic defined by the author [Bull. Amer. Math. Soc. **53**, 1186–1191 (1947); these Rev. **9**, 378]. *P. O. Bell* (Lawrence, Kan.).

Lemoine, Simone. Sur l'équation fondamentale de la théorie des réseaux conjugués persistants. C. R. Acad. Sci. Paris **229**, 1117–1118 (1949).

The problem of finding surfaces which permit isometric deformations with the preservation of a conjugate net has been reduced to the solution of a fourth order partial differential equation. See, for example, V. Lalan [same C. R. **228**, 1842–1844 (1949); these Rev. **10**, 738]. This paper gives a new derivation of this equation based on the equation of Liouville. In addition there is given a geometric interpretation of the function $R(u,v)$ which appears in Lalan's formulation of this problem. *C. B. Allendoerfer.*

Lalan, Victor. Sur l'emploi d'un repère canonique dans l'étude des réseaux conjugués. C. R. Acad. Sci. Paris **229**, 1115–1116 (1949).

Given a conjugate net on a surface, the author attaches a moving frame $Me_1e_2e_3$ with e_1 and e_2 tangent to the curves of the net and e_3 a unit normal to the surface. Using this frame and the methods of É. Cartan, he derives the Gauss and Codazzi equations of the surface. The same approach is used to sketch a proof that given a ds^2 and a net, in general there exists a unique surface having this ds^2 and having this net as a conjugate net. In exceptional cases there are two such surfaces, or else infinitely many of them. *C. B. Allendoerfer* (Haverford, Pa.).

Lalan, V. Les formes minima des surfaces d'Ossian Bonnet. Bull. Soc. Math. France **77**, 102–127 (1949).

É. Cartan has considered surfaces which can be isometrically deformed with preservation of the principal curvatures [Bull. Sci. Math. (2) **66**, 55–72, 74–85 (1942); these Rev. **5**, 216]. The author calls such surfaces "surfaces of Ossian Bonnet." In this paper these surfaces are studied as a special category of surfaces of isothermal mean curvature [see Lalan, Canadian J. Math. **1**, 6–28 (1949); these Rev. **10**, 400]. A complete classification is obtained, and the two fundamental forms of each "class" and "type" are derived to within the integration of a single third order differential equation whose independent variable is the mean curvature. The chief results have already been announced [C. R. Acad. Sci. Paris **226**, 214–216, 777–779, 1950–1952 (1948); these Rev. **9**, 376, 529; **10**, 63]. *C. B. Allendoerfer.*

Lalan, Victor. Problème d'Ossian Bonnet et la théorie de l'immersion d'un ds^2. Ann. Sci. École Norm. Sup. (3) **66**, 95–124 (1949).

The problem of Bonnet is the determination of those surfaces which can be isometrically deformed with preservation of the principal curvatures. For a general bibliography on this problem see É. Cartan [Bull. Sci. Math. (2) **66**, 55–72, 74–85 (1942); these Rev. **5**, 216]. The present paper is a reconsideration of this theory in terms of the author's recent work on the immersion of surfaces and on surfaces of isothermal mean curvature [Bull. Soc. Math. France **75**, 63–88 (1947); **76**, 20–48 (1948); Canadian J. Math. **1**, 6–28 (1949); these Rev. **9**, 375; **10**, 568, 400]. Essentially all the results are included in these papers, but the material is reworked and coordinated. The first part of the paper considers the general problem of immersion of surfaces, and it is shown that the problem reduces to the determination of the mean curvature. This result is due to H. W. Alexander [Trans. Amer. Math. Soc. **47**, 230–253 (1940); these Rev. **1**, 269] and T. Y. Thomas [Bull. Amer. Math. Soc. **51**, 390–399 (1945); these Rev. **7**, 30]. In the second part, the author shows that surfaces of Bonnet are either surfaces of constant mean curvature or of isothermal mean curvature. He then discusses the classification and special properties of such surfaces with particular reference to the work of É. Cartan. *C. B. Allendoerfer.*

Yanenko, N. N. The geometric structure of surfaces of small type. Doklady Akad. Nauk SSSR (N.S.) **64**, 641–644 (1949). (Russian)

In every point of a surface V_m imbedded in Euclidean E_{m+q} an orthonormal basis is considered consisting of m

tangent and q normal vectors. The differentials of the tangent vectors are considered as linear combinations of tangent and normal vectors; the coefficients of normal vectors in these linear combinations play (under the name of mixed forms) an important part in the work. In terms of these forms two arithmetical invariants are introduced: one, the rank r, is the number of parameters on which the tangent planes (at all points of the surface) depend (so that in E_3 the rank of a developable surface is one); the other is essentially the Allendoerfer type [Amer. J. Math. 61, 633–644 (1939); these Rev. 1, 28]. We try now to replace the above forms by their linear combinations which have the same type. The rank of the smallest such set is called ρ. When $\rho=r$ the original set is called simple and in this special case a number N_q^t is introduced which is the maximum of r for these q and t. The precise value of N_q^t is obtained only for $q=2$, namely, $N_q^t=2t+1$. For higher values of q only inequalities are obtained, namely

$$N_q^0\leqq\tfrac{1}{2}q(q-1),\quad N_q^1<q^3+q^2,\quad N_q^2<\tfrac{1}{2}q(q^2+q+2).$$

In the general case V_m is a subspace of V_{m+s} $(s<q)$ whose rank is $(N_q^t+1)2^{q-s+\rho}$. For $q=2$ more precise results are obtained: either V_m is of rank $2t$ or $2t+1$, or it is a subspace of a hypersurface V_{m+1} of rank $r=t$. Finally, we have a theorem: a necessary condition for non-unique imbedding of V_m in a E_{m+q} is that V_m allows a stratification into an R-parametric family of metrics V_{m-R}, each of class $s=q-1$. Again for $q=2$ we have more precisely: V_m is imbeddable in E_{m+2} in essentially different ways only when V_m allows a stratification into a 5-parametric family of Euclidean metrics, or when V_m can be stratified into a two-parameter family of metrics of class one. It seems likely that the author is unaware of Chern's paper [J. Indian Math. Soc. (N.S.) 8, 29–36 (1944); these Rev. 6, 216] which gives a modified definition of type somewhat more convenient for his purposes than that of Allendoerfer. *G. Y. Rainich.*

Yanenko, N. N. On some necessary conditions for the deformability of surfaces in a multidimensional Euclidean space. Doklady Akad. Nauk SSSR (N.S.) 65, 449–452 (1949). (Russian)

This is a continuation of the preceding article. Several new concepts are introduced. If a V_{m+1} is isometrically transformed we say that a V_m which is contained in it undergoes a co-bending. An isometric deformation which is not a co-bending is called proper. A rank of a system of forms was defined in the preceding article; now the rank of a bending is introduced as the rank of the system of mixed forms of the original surface and of those of the deformed surface. Using this terminology an earlier theorem is strengthened: in the case $q=2$ the rank of a proper bending cannot exceed 4. If in a bending of a V_m a subsurface V_{m-p} goes into one congruent to itself, then V_{m-p} is called an element of congruency of V_m. With this terminology we can say: if V_m in E_{m+2} undergoes a co-bending the elements of congruency are surfaces of class one and of dimension at least $m-2$; if it undergoes a proper bending the elements of congruency are planes of dimension at least 4. Consider a V_m in E_{m+2} which consists of an r-parameter family of planes so that along each of these planes the normal vector remains constant; then a direction in V_m is called focal if two consecutive planes in that direction intersect in a plane of the next smaller dimension. A one-parameter family of planes is called a focal family if the direction determined by the family is focal. Using this concept we can say: a necessary condition for V_m in E_{m+2} to permit a proper bending is that it can be stratified into ∞^{m-r} focal families of planes E_{m-r} with $r\leqq 4$. The paper concludes with the consideration of extreme cases: one in which every direction is focal, and another in which there are only four linearly independent focal directions. *G. Y. Rainich.*

Havlíček, Karel. Sur les surfaces enveloppes de sphères. Časopis Pěst. Mat. Fys. 74, 21–40 (1949). (French. Czech summary)

Let $\overset{(i)}{b}_{\lambda\mu}$ $(i=2, 3)$ be the second and third fundamental tensor of a surface S and let the rank of $\overset{(i)}{b}_{\lambda\mu}$ be 2 so that $\overset{(i)}{b}_{\lambda\mu}\overset{(i)}{P}{}^{\mu\nu}=\delta_\lambda^\nu$ admits only one solution $\overset{(i)}{P}{}^{\mu\nu}$ $(i=2$ or $i=3)$. Introduce a tensor

$$(1)\qquad \overset{(i)}{u}_{\omega\mu\lambda}=b_{\omega\mu\lambda}-\tfrac{3}{4}\overset{(i)}{Q}_{(\omega}\overset{(i)}{b}_{\mu\lambda)}$$

(where $b_{\omega\mu\lambda}=b_{(\omega\mu\lambda)}$ is the covariant derivative of $\overset{(2)}{b}_{\lambda\mu}$ and $\overset{(i)}{Q}_\omega=b_{\omega\mu\lambda}\overset{(i)}{P}{}^{\mu\lambda}$) and the scalar

$$(2)\qquad \overset{(i)}{S}\equiv\overset{(i)}{u}_{\omega\mu\lambda}\overset{(i)}{u}_{\alpha\beta\gamma}\overset{(i)}{P}{}^{\omega\alpha}\overset{(i)}{P}{}^{\mu\beta}\overset{(i)}{P}{}^{\lambda\gamma},\qquad i=2, 3.$$

Let any ruled surface (any surface-envelope of spheres) be called a surface of type (1) (of type (2)).

A necessary and sufficient condition that S be of the type (i) is $\overset{(i)}{S}=0$ $(i=2$ or $i=3)$. The case $i=1$ is known [Hlavatý, Differentialgeometrie der Kurven und Flächen und Tensorrechnung, Noordhoff, Groningen-Batavia, 1939, pp. 435–440; these Rev. 1, 27]. The case $i=2$ is the main result of the paper under review. Its proof (similar to the case $i=1$ [loc. cit.] with the changes suggested by Lie sphere-line transformation) is based on a tensor $v_{\omega\mu\lambda}$ [which turns out to be $-b_{\omega\mu\lambda}$: reviewer's remark]. In the second part the author shows how the scalars $\overset{(i)}{S}$ may be constructed simultaneously in the five-dimensional space of Plücker's line coordinates by using the Lie sphere-line transformation. *V. Hlavatý* (Bloomington, Ind.).

Simonart, Fernand, et Alardin, Félix. Sur une classe de congruences R associées aux surfaces harmoniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 602–613 (1949).

Dans une étude antérieure [Bull. Sci. Math. (2) 68, 60–72 (1944); ces Rev. 7, 78], P. Vincensini a considéré les congruences H obtenues en rapportant une surface harmonique quelconque S à un plan de base π, et en menant, par la projection m d'un point quelconque M de S sur π la parallèle à la normale MN en M à S. Les congruences ainsi obtenues appartiennent au type de Ribaucour et sont en relation simple avec la représentation conforme la plus générale du plan π. Les auteurs apportent une contribution à ce subjet en remplaçant, dans la construction des congruences H, la normale MN par la perpendiculaire commune aux normales correspondantes MN et $M'N'$ à S et à sa conjuguée S'. Soient $z=H(x, y)$, $z'=H'(x, y)$ deux fonctions harmoniques associées définissant deux surfaces H et H'. Si a, b, 1 sont les paramètres directeurs de la perpendiculaire commune P aux normales à H et H' en deux points M et M' de même projection $m(x, y)$ sur xOy, on a les relations $a_x=b_y$, $a_y=-b_x$, d'où l'on déduit aisément l'équation différentielle des développables de la congruence $[P]$: $a_y(dx^2+dy^2)=0$, et l'expression de la côte $\zeta(x, y)$ du centre $\bar{M}$ du rayon générateur P de $[P]$: $\zeta=-a_y/a_x^2+a_y^2$, cette dernière expression. montrant que la surface moyenne S de $[P]$ est harmonique, Si $a_y=0$, $[P]$ est formée de droites issues d'un point fixe

et la surface harmonique H associée

$$z=\alpha \tan^{-1}(y/x)+\beta \log (x^2+y^2)^{\frac{1}{2}}+\gamma$$

résulte d'une composition linéaire de l'hélicoïde minima réglé et de sa conjuguée. Dans le cas general $a_y\neq 0$ les développables de [P] correspondent aux deux familles de droites isotropes du plan xOy, et, en prenant comme nouvelles variables les paramètres de ces deux familles, les auteurs établissent que [P] appartient à la famille des congruences de Ribaucour, puis que la surface génératrice de [P] est un paraboloïde (P) de révolution d'axe Oz, enfin que les projections m et m' du point M de S et du point homologue $\bar{M}'$ de (P) se correspondent dans une correspondance conforme directe du plan xOy, correspondance qui est d'ailleurs la plus générale de cette espèce. En ce qui concerne ce dernier résultat, et reprenant une construction indiquée dans le mémoire cité au début de cette analyse, les auteurs arrivent à la conclusion suivante. Soit Σ une surface arbitraire, I un point fixe de l'espace, m la projection d'un point quelconque M de Σ sur xOy et m' le point où la parallèle menée par I à la normale en M à Σ perce xOy; pour que la correspondance (m, m') soit conforme il faut et il suffit: ou bien que Σ soit harmonique et alors (m, m') est inverse, ou bien que Σ soit un paraboloïde de révolution auquel cas (m, m') est directe. Eu égard aux congruences [P], il résulte de ce qui précède que, si l'on mène par un point fixe I une parallèle δ à l'un quelconque P des rayons de [P], les points de percée des deux droites P et δ déterminent, dans le plan xOy, la correspondance conforme directe la plus générale. *P. Vincensini* (Marseille).

Padma, N. Skew tensors and region-complexes in S_n. J. Indian Math. Soc. (N.S.) **13**, 91–104 (1949).

Let $p^{i_1\cdots i_{r+1}}$ be a simple $(r+1)$-vector. Then $\rho p^{i_1\cdots i_{r+1}}$ $(\rho\neq 0)$ may be thought of as homogeneous Plücker coordinates of a linear space S_r in a projective linear space S_n $(r<n)$. A complex Γ_r of these S_r is defined by $p^{i_1\cdots i_{r+1}}w_{i_1\cdots i_{r+1}}=0$, where $w_{i_1\cdots i_{r+1}}$ is a covariant $(r+1)$-vector (not necessarily simple). A singular point of Γ_r is a point such that all r-dimensional spaces incident with it belong to Γ_r. If x^i is an arbitrary point of S_n then $x^{i_{r+1}}w_{i_1\cdots i_{r+1}}$ ("an x-contract of Γ_r") defines a Γ_{r-1}. The paper is devoted chiefly to the study of singular points, x-contracts (and their dual objects). Theorems of the following types are proved. The join of two singular points of a Γ_r is composed entirely of singular points (and dually). The x-contract Γ_{r-1} of a Γ_r is the totality of all S_{r-1} whose join with x belongs to Γ_r (and dually). The point x is always a singular point of the x-contract Γ_{r-1} of Γ_r (and dually). Also the geometrical interpretation of products of the type $w_{[i_1\cdots i_{r+1}}p_{j_1]\cdots j_r}=0$ is given. *V. Hlavatý* (Bloomington, Ind.).

Pantazi, Al. Sur les variétés non-holonomes à asymptotiques rectilignes et leur déformation projective. Bull. Math. Soc. Roumaine Sci. **48**, 3–26 (1947).

[There are two pages numbered 17 and two numbered 18.] This paper completes the study of projective deformation of nonholonomic surfaces in a three-space which the author carried out in an earlier paper [same Bull. **45**, 33–47 (1943); these Rev. **7**, 33]. It is concerned with the case that the asymptotics are straight lines. For such nonholonomic surfaces a projective deformation can be characterized as a correspondence which preserves the asymptotics and the principal curvature. The author determined all these surfaces which have a constant principal curvature and which admit an eight-parameter group of projective deformations into themselves. *S. Chern* (Chicago, Ill.).

Smirnov, R. V. p-conjugate systems. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 162–163 (1949). (Russian)

This note deals with p-conjugate systems in n-dimensional projective space. By such a system is understood a set of $p(p-1)$-dimensional surfaces in V_n such that the intersections of any $p-2$ of them lie in a two-dimensional surface, the remaining two cutting out of this two-dimensional surface a conjugate net. The differential equations defining such a p-system admit solutions depending on $p(p-1)$ arbitrary functions of two variables.

M. S. Knebelman (Pullman, Wash.).

Kanitani, Joyo. On a generalization of the projective deformation. Mem. Coll. Sci. Univ. Kyoto Ser. A. **25**, 23–26 (1947).

Let R_2 and R_2' be two-dimensional spaces with projective connection. These spaces will be said to be projectively deformable into each other if we can realize a one-to-one correspondence between their points in the following manner. Let M and M' be any two corresponding points on R_2 and R_2', respectively. Develop any homologous curves L and L' passing through M and M', respectively, taking a C on S_2 as the common image of M and M', and giving at this point a common initial position to the moving frames of reference which move along the developments of L and L', respectively. Take, in the vicinity of C, two homologous points P and Q on these developments. Then the écart PQ is an infinitesimal of the third order at least with respect to the écart CP. The spaces R_2, R_2' can be immersed in an S_4 so that they become ruled surfaces V_2, V_2'. The author gives a property of the correspondence between these ruled surfaces. *M. Haimovici* (Jassy).

Cossu, Aldo. Trasformazioni conformi in una coppia di punti corrispondenti e nei punti dei loro intorni del primo ordine. Boll. Un. Mat. Ital. (3) **4**, 122–127 (1949).

Continuation of a paper by the reviewer [Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 138–151 (1942); these Rev. **8**, 349] on conformal transformations between two Euclidean planes. If a point-transformation is conformal in a pair $(O, \bar{O})$ of corresponding points and in two infinitely near pairs of points, then it is conformal at all points in the neighborhood of the first order of the given pair. In this case, the correspondence between the centers of curvature of corresponding curvilinear elements E_2 is the projectivity determined by the characteristic projectivities on the isotropic inflexional lines through the two points $O, \bar{O}$. Extension to hyperspaces. *E. Bompiani* (Rome).

Levine, Jack. Fields of parallel vectors in conformally flat spaces. Duke Math. J. **17**, 15–20 (1950).

A $C_n(p, q)$ $(n>3)$ denotes a conformally flat space which admits p fields $\underset{u}{\lambda}^i$ $(u=1, \cdots, p)$ of parallel non-null vectors and q fields $\underset{v}{\lambda}^i$ $(v=1, \cdots, q)$ of parallel null vectors. (If $p+q>0$ then $C_n(p, q)$ is a flat space.) Because we deal with a conformally flat space the integrability conditions of

$$(1)\qquad \underset{A}{\lambda}{}_{,j}^{i}=0, \qquad A=1, \cdots, p+q,$$

may be written

$$(2)\qquad \underset{A}{\lambda}{}_j T_k^h-\underset{A}{\lambda}{}_k T_j^h=0 \qquad (T_j^i=\delta_j^i R/(n-1)-R_j^i)$$

or

$$T_j^i = \underset{A}{\lambda}_j \underset{A}{T}^i = \underset{B}{\lambda}_j \underset{B}{T}^i, \tag{3}$$

and this formula shows (for a non-flat C_n) that the rank of the matrix $((\underset{A}{\lambda}_j))$ is equal to one. Hence we have either $p=1$, $q=0$ or $p=0$, $q=1$. If we put $\underset{A}{\lambda}_j = \underset{A}{\mu}\lambda_j$ then from (3): (4) $T_{ij}=\lambda_i\lambda_j S$, where $S\lambda^i\lambda^j g_{ij}=R/(n-1)$ and R is 0 or not 0 for $p=0$, $q=1$ or for $p=1$, $q=0$. Then the necessary and sufficient conditions for $r=p+q=1$ are

$$T_{i[j}T_{h]k}=0, \quad R_{hijk,m}=R_{hijk}t_{,m},$$

where $T_{ij}\equiv g_{ij}R/(n-1)-R_{ij}\neq 0$. For $t_{,m}=0$ and $R\neq 0$ we have $p=1$, $q=0$. This main result enables the author to find new canonical forms of the metric as well as the equation of embedding of a $C_n(1, 0)$ in a flat S_{n+1}.

V. Hlavatý (Bloomington, Ind.).

Yano, Kentaro. Bemerkungen über infinitesimale Deformationen eines Raumes. Proc. Japan Acad. **21** (1945), 171–178 (1949).

Yano, Kentaro. Sur la déformation infinitésimale des sous-espaces dans un espace affine. Proc. Japan Acad. **21** (1945), 248–260 (1949).

Yano, Kentaro. Sur la déformation infinitésimale tangentielle d'un sous-espace. Proc. Japan Acad. **21** (1945), 261–268 (1949).

Yano, Kentaro. Quelques remarques sur un article de M. N. Coburn intitulé "A characterization of Schouten's and Hayden's deformation methods." Proc. Japan Acad. **21** (1945), 330–336 (1949).

Yano, Kentaro. Lie derivatives in general space of paths. Proc. Japan Acad. **21** (1945), 363–371 (1949).

[For the article referred to in the 4th paper see J. London Math. Soc. **15**, 123–136 (1940); these Rev. **2**, 163, 419.] This series of papers is a résumé of results already known in this field. The definition of Lie derivation [J. A. Schouten and D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, v. I, Noordhoff, Groningen-Batavia, 1935, p. 150] is a modification of the one usually given, and so is the deduction of some of the formulae. The author is not aware of papers which have appeared since 1945 on this subject, such as those by E. T. Davies [Quart. J. Math., Oxford Ser. **16**, 22–30 (1945); these Rev. **7**, 81], R. S. Clark [Proc. Cambridge Philos. Soc. **41**, 210–223 (1945); these Rev. **7**, 125], and B. Su [Trans. Amer. Math. Soc. **61**, 495–507 (1947); these Rev. **8**, 602].

E. T. Davies.

Yano, Kentaro. On the fundamental differential equations of flat projective geometry. Proc. Japan Acad. **21** (1945), 392–400 (1949).

Let the ξ^i denote coordinates in an n-dimensional projective flat space P_n. It is convenient to introduce in P_n supernumerary coordinates, which can be established in different ways. The first possibility is to introduce a ξ^0 with the transformations: $\xi^{0'}=\xi^0+\log\rho(\xi^i)$; $\xi^{i'}=f^{i'}(\xi^i)$ [Veblen, J. London Math. Soc. **4**, 140–160 (1929)]. Then the coordinates $Z^\lambda(e^{\xi^0}, e^{\xi^0}\xi^i)$ satisfy the differential equations $\partial_\mu\partial_\lambda Z+\Gamma^\kappa_{\mu\lambda}\partial_\kappa Z=0$ ($\partial_\lambda=\partial/\partial\xi^\lambda$). The $\Gamma^\kappa_{\mu\lambda}$ define a projective connexion. Another possibility is to introduce homogeneous coordinates x^κ with the transformations $x^{\kappa'}=\varphi^{\kappa'}(x^\lambda)$, the functions $\varphi^{\kappa'}$ being homogeneous of degree 1. This leads to the parameters $\Pi^\kappa_{\mu\lambda}$ homogeneous of degree -1 [Schouten and Haantjes, Compositio Math. **3**, 1–51 (1936)]. The author gives the relation between the components $\Pi^\kappa_{\mu\lambda}$ and $\Gamma^\kappa_{\mu\lambda}$. The differential equations of paths are given for both connexions.

J. Haantjes (Leiden).

Yano, Kentaro. On the flat conformal differential geometry. I. Proc. Japan Acad. **21** (1945), 419–429 (1949).

Yano, Kentaro. On the flat conformal differential geometry. II. Proc. Japan Acad. **21** (1945), 454–465 (1949).

Let $\xi^\nu(\nu, \lambda, \mu=1, \cdots, n)$ be (curvilinear) coordinates of a flat conformal space C_n and let $a^\Sigma=a^\Sigma(\xi)$ be the homogeneous coordinates of its elements $\mathbf{a}(\Sigma, \Lambda, \Omega=0, \cdots, n, \infty)$. If either $\mathbf{a}\cdot\mathbf{a}\equiv a^1a^1+\cdots+a^na^n-a^0a^\infty=0$ or $\mathbf{a}\cdot\mathbf{a}\neq 0$ then $\mathbf{a}$ is either a point (point-hypersphere) or a hypersphere in C_n. Let $\mathbf{a}_0$ be a generic point of C_n so that (1) $\mathbf{a}_0\cdot\mathbf{a}_0=0$. Hence the n hyperspheres $\mathbf{a}_\lambda\equiv\partial\mathbf{a}_0/\partial\xi^\lambda$ go through $\mathbf{a}_0$ and consequently they go through another point, say $\mathbf{a}_\infty$, whose factor of proportionality will be normed by $\mathbf{a}_0\cdot\mathbf{a}_\infty=-1$. The elements $\mathbf{a}_0$, $\mathbf{a}_\lambda$ being supposed linearly independent the determinant of $g_{\lambda\mu}=\mathbf{a}_\lambda\cdot\mathbf{a}_\mu$ is different from zero. The fundamental equations may be written

$$\frac{\partial}{\partial\xi^\mu}\mathbf{a}_\Sigma=\Pi^\Omega_{\Sigma\mu}\mathbf{a}_\Omega, \tag{2a}$$

whereas it may easily be shown that

$$\Pi^\Omega_{0\mu}=\delta_\mu{}^\Omega, \quad \Pi^\nu_{\lambda\mu}=\{{}^{\,\nu}_{\lambda\mu}\}, \quad \Pi^\infty_{\lambda\mu}=g_{\lambda\mu} \tag{2b}$$

and $\Pi^0_{\lambda\mu}$ is a quadratic tensor. (The Christoffel symbols { } belong to $g_{\lambda\mu}$.) The integrability conditions of (2) obtained in the usual way are (3) $I^\Omega_{\Sigma\omega\mu}=0$. They determine the tensor $\Pi^0_{\omega\mu}$. Moreover (3), for $\Omega=\infty$, is an identity, $I^\nu_{\lambda\omega\mu}$ is the conformal curvature tensor [Weyl] and $-(n-3)I^0_{\mu\nu\omega}$ is equal to the covariant divergence of $I^\nu_{\lambda\omega\mu}$. In spite of the fact that the $\Pi^\Omega_{\Sigma\mu}$ are not invariant with respect to the change of the factor of proportionality (4) $^*\mathbf{a}_0=\rho\mathbf{a}_0$ the equations (3) are invariant with respect to (4). The requirement that the determinant $^*g$ of $^*g_{\lambda\mu}={}^*\mathbf{a}_\lambda\cdot{}^*\mathbf{a}_\mu$ be equal to one amounts to (5) $\rho=g^{-1/2n}$. The corresponding $^*\Pi^\Omega_{\Sigma\mu}$ (involved in the fundamental "starred" equations (2ab)) are the coefficients of T. Y. Thomas, while in the corresponding conditions of integrability $^*I^\nu_{\lambda\omega\mu}=I^\nu_{\lambda\omega\mu}$ and for $n=3$ $^*I^0_{\lambda\omega\mu}$ is the "conformal covariant" (tensor) of J. M. Thomas. In the last section two sets of conformally invariant "Frenet formulae" for a curve are found. The first set is based on a projective parameter while the second set is essentially the same as obtained by the author in one of his previous papers [Proc. Phys.-Math. Soc. Japan (3) **22**, 595–621 (1940); these Rev. **2**, 165].

V. Hlavatý.

Sasaki, Shigeo, and Yano, Kentarô. On the structure of spaces with normal projective connexions whose groups of holonomy fix a hyperquadric or a quadric of $(N-2)$-dimension. Tôhoku Math. J. (2) **1**, 31–39 (1949).

Verff. betrachten einen Raum P_n dessen Zusammenhang projektiv-normal ist. Unter Voraussetzung, dass die Holonomiegruppe aus denjenigen projektiven Transformationen besteht, die eine nicht ausgeartete Hyperfläche zweiter Ordnung invariant lässt, kann bewiesen werden, dass der P_n projektiv zu einem Einsteinschen Raum mit nicht verschwindendem Krümmungsskalar ist. Letzteres heisst ausführlicher, dass der Einsteinsche Raum in die Klasse derjenigen Räume gehört, deren affine Übertragungen durch projektive Abänderung auseinander hervorgehen. Die Umkehrung obiger Behauptung wird ebenfalls als richtig gefunden. Für die Metrik eines Einsteinschen Raumes kann nun eine Darstellung gefunden werden, die zur Cayley-Kleinschen völlig analog ist. Nach dem vorangehenden kann dem Einsteinschen Raum ein Raum P_n zugeordnet werden,

dessen Holonomiegruppe eine ovale oder nullteilige Hyperfläche invariant lässt. Für den Abstand s zweier Punkte A, B ergibt sich, jenachdem ob der Krümmungsskalar R positiv oder negativ ist, $s=-i\{(n-1)n/4R\}^{\frac{1}{2}}\log d$ bzw. $s=\{-(n-1)n/4R\}^{\frac{1}{2}}\log d$, wobei $d=(A, B, X, Y)$ ist und X, Y die Schnittpunkte des Bildes der geodätischen Linie auf der Hyperebene der invarianten Fläche zweiter Ordnung mit dieser Fläche bedeuten. Die bisherigen Resultate sind Gegenstücke der klassischen nichteuklidischen Geometrie. Verf. untersuchen nun auch das Analogon zur Euklidischen Geometrie. Dazu muss eine Hyperebene und in ihr eine $(n-2)$-dimensionale Fläche zweiter Ordnung untersucht werden. Dadurch kommen, die bisher ausgeschlossenen, Einsteinschen Räume mit verschwindendem Krümmungsskalar zum Vorschein. In dieser Richtung finden Verff. folgendes Resultat: Ein P_n mit projektiv-normalen Zusammenhang dessen Holonomiegruppe eine $(n-2)$-dimensionale Fläche zweiter Ordnung einer Hyperebene invariant lässt, gehört in diejenige Klasse von aufeinander bahntreu abbildbaren affinzusammenhängenden Räumen, die insbesondere einen solchen Weylschen Raum enthalten, für den $R_{ij}=0$ gilt. Dabei ist R_{ij} in bekannter Weise durch Verjüngung aus dem Krümmungsaffinor herleitbar. Weylsche Räume dieser Art enthalten die Einsteinschen Räume verschwindenden Krümmungsskalars als Sonderfall. Es gilt auch die Umkehrung des vorangehenden Ergebnisses.

O. Varga (Debrecen).

Iwamoto, Hideyuki. Sur les espaces Riemanniens symétriques. I. Jap. J. Math. **19**, 513–523 (1948).

The author's object is to investigate the symmetric Riemannian spaces of compact simple or semi-simple groups. The first section of the paper concerns a real Grassmann manifold $R_{n,k}$, that is, the totality of k-planes passing through the origin of a Euclidean n-space. If M_0 is a k-plane of $R_{n,k}$, there is a subgroup g of $O^+(n)$ that leaves M_0 invariant. The manifold $R_{n,k}$ is symmetric in Cartan's sense and admits one and only one Riemannian metric invariant under $OL(n)$. A geodesic always passes through two elements M, N of $R_{n,k}$, and an arbitrary geodesic is not closed. The number of differential invariants, or the pth Betti number, is equal to the number of ways of partitioning the integer p according to a certain rule. The manifold $R_{n,k}$ is an Einstein space. In the second and third sections similar results are given for "complex-unitary" and "complex-real" Grassmann manifolds, of which the background groups are respectively unitary-symplectic and orthogonal-symplectic. The last section of the paper is concerned with "generalized Euclidean spaces" of $\aleph_1$ and $\aleph_2$ dimensions.

See the note to p. 241, Tsuji. *H. S. Ruse* (Leeds).

Sun, J. Tseying. A note on the isometric correspondence of Riemann spaces. Duke Math. J. **16**, 571–573 (1949).

If in a Riemannian space (in the small) there are $p\leqq n$ functionally independent scalar invariants, all other scalar invariants being functionally dependent upon them, then the space is said to be of category p. T. Y. Thomas [Proc. Nat. Acad. Sci. U. S. A. **31**, 306–310 (1945); these Rev. **7**, 80] has set up scalar relationships for spaces of the same category which imply that they are isometric, and has solved the problem for n-spaces of category n and for 2-spaces of categories 2, 1, 0. In the present paper the author considers n-spaces of category $n-1$. Denoting by I_i a fundamental set of invariants for such a space, R_n, he writes $I_{jk}=g^{\alpha\beta}I_{j,\alpha}I_{k,\beta}$, $J_i=g^{\alpha\beta}I_{i,\alpha\beta}$ (Greek indices running from 1 to n and Latin from 1 to $n-1$, commas denoting covariant derivatives), and proves the following. Lemma: R_n admits coordinates y^α (covering any point P of R_n) for which $y^i=I_i$ and

$$ds^2=p_{ij}(y^1, \cdots, y^{n-1})dy^idy^j+p_{nn}(y^1, \cdots, y^{n-1})(dy^n)^2.$$

Theorem: a necessary and sufficient condition that R_n can be put in isometric correspondence with another space R_n^* (also of category $n-1$) is that the scalar equations $I_i=I_i^*$, $I_{jk}=I_{jk}^*$, $J_i=J_i^*$ admit a solution $x^\alpha=x^{*\alpha}(x^1, \cdots, x^n)$ relating the coordinates x^α, $x^{*\alpha}$ of the two spaces.

H. S. Ruse (Leeds).

Schmidt, Hermann. Winkeltreue und Streckentreue bei konformer Abbildung Riemannscher Räume. Math. Z. **51**, 700–701 (1949).

The author gives another proof of the theorem that a mapping between two Riemann spaces which preserves angle must magnify all infinitesimal arc lengths emanating from the same point in the same ratio. *A. Fialkow.*

Sen, R. N. Parallel displacement and scalar product of vectors. III. Bull. Calcutta Math. Soc. **41**, 113–120 (1949).

In two previous papers [Proc. Nat. Inst. Sci. India **14**, 45–52 (1948); same Bull. **41**, 41–46 (1949); these Rev. **10**, 479; **11**, 133], various affine connections related to an arbitrary affine connection were defined in a Riemann space and the corresponding parallel displacements of vectors were studied. The present paper continues these investigations, deriving a number of equations which are satisfied by the curvature tensors constructed from the Christoffel symbols of the space and these affine connections. *A. Fialkow.*

Blum, Richard. Sur la classe des variétées riemanniennes. Bull. Math. Soc. Roumaine Sci. **48**, 88–101 (1947).

If a Riemann space V_n is to be of class ν, quantities $b_{\alpha|ij}$ and $t_{\alpha\beta|i}$, $\alpha, \beta=1, 2, \cdots, \nu$; $i, j=1, 2, \cdots, n$, must exist such that the Gauss, Codazzi and Ricci-Kühne equations are satisfied. If quantities $b_{\alpha|ij}$, $t_{\alpha\beta|i}$ exist which satisfy the Ricci-Kühne equations, then the author enumerates the number of Gauss and Codazzi equations which must still be satisfied if V_n is to be of class $k\leqq\nu$. Similarly, if the Codazzi equations are satisfied, then the author enumerates the number of Gauss and Ricci-Kühne equations which remain to be satisfied. The results depend upon n, ν and the ranks of certain matrices. Special consideration is given to the case $\nu=1$. *A. Fialkow* (Brooklyn, N. Y.).

Graeub, Werner. Geometrische Deutung des Krümmungstensors. Arch. Math. **2**, 148–151 (1950).

Let $\Gamma^\nu_{\lambda\mu}$ be the coefficients of a (not necessarily symmetric) connection of a space L_n referred to a coordinate system x^ν. Consider a surface $x^\nu(u, v)$ in L_n and a closed Jordan curve $C(t)$ (with continuous curvatures) in the (u, v)-plane. If $\xi^\nu(t)$ is a field of parallel vectors along C and f the area enclosed by C then the difference $\Delta\xi^\nu=\xi^\nu(l)-\xi^\nu(0)$ (l being the length of C) satisfies the known formula

$$\lim_{f\to 0}\frac{\Delta\xi^\nu}{f}=R^\nu_{\omega\lambda\mu}\frac{\partial x^\mu}{\partial u}\frac{\partial x^\lambda}{\partial v}\xi^\omega.$$

The author proves it by using an inequality for $|\xi^\lambda(t)-\xi^\lambda(0)|$ and the Gauss theorem for

$$\int\left(\frac{\partial\Gamma^\nu_{\alpha\sigma}}{\partial x^\beta}-\Gamma^\nu_{\rho\sigma}\Gamma^\rho_{\alpha\beta}\right)x^\sigma\left(\frac{dx^\beta}{du}du+\frac{dx^\sigma}{dv}dv\right).$$

V. Hlavatý (Bloomington, Ind.).

Schouten, J. A. Lie's differential operator. Math. Centrum Amsterdam. Rapport **ZW-1949-010**, 7 pp. (1949). (Dutch)

This is a résumé of a lecture. Let φ be a field of objects in X_n. The covariant differential of φ with respect to a symmetric linear connection in X_n and Lie's differential of φ with respect to an infinitesimal transformation in X_n are defined. Applications to the geodesics and the minimal surfaces in a Riemannian space are given.

W. van der Kulk (Providence, R. I.).

Gheorghiu, Octavian Emil. Les lois de transformation des objets géométriques spéciaux linéaires de classe ν, avec une composante, en X_1. C. R. Acad. Sci. Paris **229**, 611–613 (1949).

The author gives all the geometric objects with one component in a one-dimensional space X_1 with the restriction that the transformation is linear: $\Omega_2=A\Omega_1+B$. Here A and B may depend explicitly on the coordinates ξ^1 and $\xi^{1'}=\varphi(\xi^1)$. His results are in agreement with those of Gołąb [Ann. Soc. Polon. Math. **19** (1946), 7–35 (1947); these Rev. **9**, 206] and Pensov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 563–566 (1946); these Rev. **9**, 67] who considered only pure objects but without the restriction mentioned above.

J. Haantjes (Leiden).

***Hachtroudi, Mohsen. Les connexions normales affines et weyliennes.** Université de Téhéran. Faculté des Sciences. Publ. no. 43, 1948. iv+214 pp.

Ce volume se trouve dans la tradition des travaux de É. Cartan sur la recherche des invariants d'une équation différentiel du second ordre relativement aux transformations ponctuelles. Le principal problème traité ici est la recherche des invariants des systèmes différentiels relativement aux transformations ponctuelles à jacobien unitaire. On peut attacher alors au système, d'une manière invariante par rapport aux transformations envisagées, un espace d'éléments linéaires à connexion affine. Cette connexion affine conserve les hypervolumes de l'espace, est sans torsion et sans courbure tangentielle. L'auteur traite également d'une manière détaillée le problème analogue pour les transformations ponctuelles conformes dans les cas où le nombre de dimensions n est 2, 3 ou supérieur à 3. Il s'introduit alors des connexions de Weyl normales isométriques, c'est-à-dire admettant une mesure relative locale invariante pour les différentes directions issues d'un même pointe. L'auteur approfondit, grâce à ces connexions, différents problèmes de géométrie différentielle locale. La rédaction de ce livre est clair et sa lecture peut conduire à se familiariser avec les méthodes de géométrie différentielle de É. Cartan.

A. Lichnerowicz (Paris).

Davies, E. T. On the second variation of a simple integral with movable end points. J. London Math. Soc. **24**, 241–247 (1949).

A formula for the second variation of the length integral is derived in a geometry based upon a vector density of weight p, which includes both Finsler geometry and Cartan geometry. *S. Chern* (Chicago, Ill.).

Busemann, Herbert. The geometry of Finsler spaces. Bull. Amer. Math. Soc. **56**, 5–16 (1950).

The paper gives a report of recent results and viewpoints, almost all due to the author, on a geometrical theory of Finsler spaces. It proceeds with discussions of the historical background, motivation of the problems, and illustrative examples, technical details being omitted. The approach to the theory is entirely different from the classical one, in which the space is considered as a space of line elements and emphasis is laid on the connection. By looking at the problems geometrically, it seems that more satisfactory generalizations of the notions of Riemannian geometry are possible. Since Finsler space is locally Minkowskian (or finitely dimensional Banach), it is clear that Minkowski geometry and convex bodies play important rôles. This point is illustrated by various examples. In particular, discussions are given of measures in Minkowski geometry, the isoperimetric problem, and relative differential geometry.

S. Chern (Chicago, Ill.).

***Synge, J. L., and Schild, A. Tensor Calculus.** Mathematical Expositions, no. 5. University of Toronto Press, Toronto, Ont., 1949. xii+324 pp. $6.00.

The authors state in the preface that this book is intended as a general brief introduction to tensor calculus, without claim to be exhaustive in any particular direction. It includes a chapter on the applications of tensor analysis to classical dynamics, and another upon its applications to hydrodynamics, elasticity and electromagnetic radiation. Relativity is not dealt with at length because the authors felt it to be adequately covered in many other treatises.

The book begins in an elementary way, an N-dimensional space being presented as a continuum of points defined by ordered sets of N real numbers (x^r) subject to transformations of nonvanishing Jacobian. The differentials dx^r are taken as the prototype of a contravariant vector, and the gradient of a scalar as that of a covariant vector, tensors then being defined in terms of their laws of transformation. The geometry is almost entirely in the small, though references are made from time to time to properties of Riemannian manifolds in the large. The first of these references is made in connection with geodesic deviation, for which the Levi-Civita formula is obtained. Another and more extended reference appears in the chapter on classical dynamics, a section of which is devoted to the topology of the configuration-spaces of particular dynamical systems.

Riemannian geometry is introduced by analogy with classical differential geometry, but, throughout the book, allowance is made for the possibility of an indefinite ds^2, indicators ϵ $(=\pm 1)$ being inserted wherever necessary. All the main formulae and ideas of Riemannian geometry are developed, including orthogonality, parallelism, curvature, generalized Frenet formulae, special coordinate systems and special types of space. These topics occupy the first four chapters, which are immediately followed by the chapters on physical applications referred to above.

Of the remaining two chapters of the book, the former is concerned with relative (weighted) tensors, with ideas of volume, and with Green-Stokes theorems in nonmetrical as well as in Riemannian spaces, careful account being taken of orientability. The final chapter gives an outline of the ideas of non-Riemannian geometry, with some discussion of the geometrical significance of symmetry in an affine connection, and finishes with a short account of Weyl spaces and of spaces with projective connection. Each chapter ends with a summary of formulae and a set of exercises.

Standard terminology is not always adhered to: for example, the authors attach to the phrases "normal coordinate system" and "homogeneous coordinates" meanings other than the traditional ones. There are no references to sources, but a list of eighteen standard treatises or monographs is given in a bibliography at the end of the book. Difficulties

of principle are not glossed over, but are always dealt with in a simple and clear fashion, and the general style is cheerful. Opinions will perhaps differ about some points of detail or even about the general plan of the book, but there is no doubt that it provides a refreshing and useful addition to the introductory literature on tensor calculus.

H. S. Ruse (Leeds).

Zeuli, Tino. Su di una notevole proprietà del rotore di un campo vettoriale. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 223–225 (1949).

For any vector field which is independent of time the lines along which the circulation of the vector is stationary are the vortex lines of the field. *L. M. Milne-Thomson.*

Dufresnoy, Jacques, et Revuz, André. Introduction au calcul différentiel extérieur. Bull. Tech. Univ. Istanbul 1, 48–66 (1948). (French. Turkish summary)

An expository article. *S. Chern* (Chicago, Ill.).

***Synge, J. L. Electromagnetism without metric.** Proc. Symposia Appl. Math. 2, 21–48 (1950). $3.00.

An electromagnetic field is described by a pair of antisymmetric tensor fields in a four-dimensional space. This paper is concerned with the geometry associated with such a pair of tensors as expressed by algebraic and differential conditions on the tensors. A pair of such tensors, P_{mn} and Q_{mn}, are said to be Maxwellian complements at a point (or in a domain) if there exists a system of coordinates called canonical coordinates for which the equations

$$Q_{23}=P_{14},\quad Q_{14}=-P_{23},$$
$$Q_{31}=P_{24},\quad Q_{24}=-P_{31},$$
$$Q_{12}=P_{34},\quad Q_{34}=-P_{12}$$

hold at the point in question (or throughout the domain). Necessary and sufficient conditions for two tensors to be Maxwellian complements at a point have been given by G. H. F. Gardner. In this paper they are shown to be insufficient for the existence of canonical coordinates in a domain. Sufficiency conditions for this are described as is the group of transformations which permute the canonical coordinates for all tensor pairs P_{mn}, Q_{mn} among themselves. This group is shown to be the conformal group. The paper ends with a proposed modification of the Maxwell equations in vacuo. *A. H. Taub* (Urbana, Ill.).

Rumer, Yu. B. Action as the coordinate of a space. III. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 19, 868–875 (1949). (Russian)

This paper is independent of the previous two under the same title [same vol., 86–94, 207–214 (1949); these Rev. 10, 580]. It is concerned with a mathematical problem, the representation of spinors in a general n-dimensional Riemannian space. The treatment is similar to those given earlier by various authors [for references see É. Cartan, Leçons sur la Théorie des Spineurs, Actual. Sci. Ind., nos. 643, 701, Hermann, Paris, 1938]. It is assumed that in the space there is not only a metric but also a set of n vector fields defining at each point a set of n fundamental vectors. There are two separate groups to be considered, the group of general coordinate transformations leaving the metric invariant, and the group of orthogonal transformations of the fundamental vectors at a fixed point. Spinors are defined by means of this second group. Having established definitions and standard properties of spinors, the author concludes by deriving the form of the Dirac equation in general relativity. *F. J. Dyson* (Princeton, N. J.).

NUMERICAL AND GRAPHICAL METHODS

***Schulz, Günther. Formelsammlung zur praktischen Mathematik.** Sammlung Göschen Band 1110. Walter de Gruyter & Co., Berlin, 1945. 147 pp. 2.40 DM.

This pocket-size little book contains in compact form a useful collection of constants, methods, and formulas for numerical mathematics. The scope is indicated by the contents. I, General Aids for Computers: values of constants, lists of series, estimates of error, slide rule, and nomography. II, Least Squares: direct and indirect observations and the solution of the normal equations. III, Solution of Equations: successive approximations, regula falsi, algebraic equations, and systems of equations. IV, Interpolation: the calculus of differences, the interpolating polynomial, interpolation in tables, double interpolation, numerical differentiation and integration. V, Quadrature and Summation: using the interpolating series, quadrature by summation, formulas for summation. VI, Approximation to Arbitrary Functions: orthogonal systems, harmonic analysis. VII, Integration of Differential Equations: Runge's method, repeated integrations, methods of Adams and Störmer, difference methods for both ordinary and partial differential equations. *W. E. Milne* (Corvallis, Ore.).

Freudenthal, H. Mathematical problems of feed back. Nederl. Tijdschr. Natuurkunde 15, 275–281 (1949). (Dutch)

The author discusses the analogy between stability of feed back systems and convergence of iterative processes. The discussion is elementary and contains no new results. *H. B. Curry* (State College, Pa.).

Wilkes, M. V., and Renwick, W. The EDSAC—an electronic calculating machine. J. Sci. Instruments 26, 385–391 (1949).

van der Poel, W. L. Connections of automatic calculating machines. Nederl. Tijdschr. Natuurkunde 15, 255–264 (1949). (Dutch)

This is an elementary discussion of the performance of computing operations by two position devices, such as relays. There is a brief discussion of operations in the binary system, including carries and negative numbers; of a relay connection for addition; and of memory elements. *H. B. Curry* (State College, Pa.).

***Bückner, Hans. Entwicklung der Rechengetriebetechnik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 43–48. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This is a sketchy account of a few developments in analogue computing instruments in Germany during the indicated period. Eight devices which came to the author's notice while he was attached to the Ascania Works in Berlin are described very briefly, and there is a short review of books by Meyer zur Capellen, Willers, and Kuhlenkamp. *H. B. Curry* (State College, Pa.).

Korn, Granino A. Design and construction of universal function generating potentiometers. Rev. Sci. Instruments 21, 77–81 (1950).

In analogue computers using function-generating potentiometers it is common practice to build the potentiometer

to produce a specified function. This is a rather inflexible procedure and it is also costly. The author treats the method in which a tapped linear potentiometer is connected to an external adjustable network so that the tap points can be set to the required voltage levels for any required function. At intermediate points the output voltage is interpolated almost linearly between the adjacent tap points. One unit of this type can be used to produce a wide range of functions. Design and construction data are provided.

S. H. Caldwell (Cambridge, Mass.).

McCann, G. D. The California Institute of Technology electric analog computer. Math. Tables and Other Aids to Computation **3**, 501–513 (1949).

A brief description of the composition of the machine and details of some of its components. The kinds of problems treated and the general methods used are discussed. The bibliography lists a number of papers which contain more detailed information. *S. H. Caldwell.*

***Malavard, L. Quelques récentes applications de la méthode d'analogies électriques.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 55–71. Centre National de la Recherche Scientifique, Paris, 1949.

Segerdahl, C.-O. A table of the interest intensity function for interest intervals of 0.01% from 0% to 7%. Skand. Aktuarietidskr. **32**, 15–20 (1949).

This gives a table of $\ln(1+i)$ to 9 decimals for $i=0(.0001).07$. A comparison with the New York Mathematical Tables Project's Table of Natural Logarithms, v. III [1941; these Rev. **3**, 152], which contains the logarithms of the decimal numbers from 0.0001 to 5.0000 to 16 decimals, has revealed no discrepancy. When making the comparison the undesirability of using a smaller type for figures after the decimal than before it, with the resulting difficulty in reading several numbers in succession, was very apparent. *J. C. P. Miller* (Washington, D. C.).

Hartree, D. R. The tabulation of Bessel functions for large argument. Proc. Cambridge Philos. Soc. **45**, 554–557 (1949).

When tabulating a mathematical function it is a common practice, when difficulties of interpolation arise owing to rapid variation of the function as the argument changes, to tabulate an auxiliary function of much less rapid variation. The paper now reviewed considers the compact tabulation of Bessel functions of large argument. They are oscillatory, and, for tabulation, functions of less rapid variation may be readily found, which can be tabulated satisfactorily at much wider intervals: for example, the functions

$$(2/\pi^{\frac{1}{2}})P_0(x)=x^{\frac{1}{2}}[(J_0+Y_0)\sin x+(J_0-Y_0)\cos x],$$
$$(2/\pi^{\frac{1}{2}})Q_0(x)=x^{\frac{1}{2}}[(J_0+Y_0)\cos x-(J_0-Y_0)\sin x].$$

Even with these functions, however, the extension to infinite range is troublesome and a further modification is proposed, that of using also an auxiliary argument. The tabulation of $(2/\pi^{\frac{1}{2}})P_0$ and $(2/\pi^{\frac{1}{2}})Q_0$ with argument $1/x^2$ is illustrated. An 8-decimal table is given for $1/x^2=0(0.01)0.05$, which is linear to 5 decimals and can be interpolated easily by the second difference Everett formula to full accuracy; this covers the range of x from about 4.5 to ∞. As a further illustration other auxiliary quantities $(2/\pi^{\frac{1}{2}})(P_0^2+Q_0^2)^{\frac{1}{2}}$ and $x\tan^{-1}(Q_0/P_0)$ are similarly tabulated. The extension to the tabulation of other functions is envisaged.

J. C. P. Miller (Washington, D.C.).

Greville, Thomas N. E. Subtabulation by least squares of finite differences. Bol. Inst. Brasil. Atuária, no. 2, 7–34 (1946). (Portuguese)

The author proposes a method of subtabulating a given table to one fifth of the interval by minimising the sum of the squares of the differences of a certain order, say n. This is in fact equivalent to the vanishing of the central differences of order $2n$ of the interpolates. The execution is made to depend upon the calculation of an auxiliary sequence of values from which the interpolates are obtained. Numerical tables are provided for the calculation in the cases $n=2, 3, 4, 5$. *L. M. Milne-Thomson* (Greenwich).

Coustal, René. Calcul de $\sqrt{2}$, et réflexion sur une espérance mathématique. C. R. Acad. Sci. Paris **230**, 431–432 (1950).

The decimal value of $\sqrt{2}$ is given to 1032 decimal places. The author used the binomial expansion $\sqrt{2}=a(1-2x)^{-\frac{1}{2}}$, where a is an approximation to $\sqrt{2}$ and $a^2=2-4x$. The approximation a was good enough to make x about 10^{-333}. The distribution of the digits of $\sqrt{2}$ and $1/\sqrt{2}$ is given. The reflection referred to in the title has to do with the consideration of the product of all the 1033 digits of any decimal. Since the average size of a digit is 9/2, the expected value of the product is put at $(9/2)^{1033}$ or about 10^{674}. However the probability that this product is equal to zero is $1-(9/10)^{1033}=1-10^{-47}$, approximately.

D. H. Lehmer (Berkeley, Calif.).

Couffignal, Louis. Calcul d'un quotient ou d'une racine carrée dans le système de numération binaire. C. R. Acad. Sci. Paris **229**, 488–489 (1949).

Lorenz, Paul. Herleitung der Näherungsformel von Laplace für die Binomialverteilung, ohne Grenzübergang. Z. Angew. Math. Mech. **29**, 368–374 (1949). (German. English, French and Russian summaries)

The normal approximation to the binomial distribution $b(k; n)$ is derived by first treating the central term $b(m; n)$ in the usual way by means of Stirling's formula and then expanding $\log b(k+1, n)-\log b(k, n)$ into a power series in k/n. Summing over $k=m, m+1, \cdots, r$ gives an approximation to $b(r, n)$. The idea has been used in various textbooks, but the author discusses the error term in greater detail.

W. Feller (Ithaca, N. Y.).

Pflanz, Erwin. Über ein Verfahren zur genäherten Auflösung von Gleichungen $f(x)=0$. Arch. Math. **2**, 5–9 (1949).

Wenn $f(x_1)f(x_2)<0$ ist, so bestimmt Verfasser ein Näherungspolynom $p(x)$ mit $p(x_1)=f(x_1)$ und $p(x_2)=f(x_2)$ mit der zusätzlichen Forderung, dass im Intervall (x_1, x_2) das Integral über $p(x)$ gleich dem über $f(x)$ ist.

E. Bodewig (Den Haag).

Friedman, Bernard. Note on approximating complex zeros of a polynomial. Comm. Pure Appl. Math. **2**, 195–208 (1949).

The author improves Lin's method for finding a linear or quadratic factor of a polynomial $f(x)$. Let $g(x)$ be the trial factor of $f(x)$: $f(x)=g(x)\cdot q(x)+r(x)$. Then an improved

factor $g_1(x)$ is found by dividing $f(x)$ by $q(x)$ in ascending powers of x: $f(x)=q(x)\cdot g_1(x)+r_1(x)$. The convergence is better than in Lin's method. The author adds a generalisation of synthetic division when the divisor is a quadratic function. *E. Bodewig* (The Hague).

✱Bloch, A. Solution of algebraic equations by means of an electrolytic tank. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 324.

Gonzalez del Valle, Angel, and Gomez Garcia, Juan Antonio. A preliminary design of an electronic machine for solving algebraic equations. Memorias de Matemática del Instituto "Jorge Juan," no. 6, iv+62 pp. (1948). (Spanish. English summary)

This discusses the design of certain components of a projected analogue machine for exhibiting on a cathode ray screen all the complex roots (in a certain region of the complex plane) of an algebraic equation. The essential part of the device is a network which is resonant when the $\beta+i\alpha$ of an impressed voltage $KE^{(\beta+i\alpha)t}$ is a root of the given equation. If the output of this network is directed to the control grid of a cathode ray tube and currents of varying frequencies are run through it, while at the same time the beam is directed to the point on the screen with coordinates α, β, the bright spots will show where the roots are. It is claimed that, by means of various coordinate shifting and scale changing devices, an accuracy of five significant figures can be obtained over a considerable range, using fewer than 50 tubes. The network is said to have been described in a report submitted to the Consejo Superior de Investigaciones Científicas in 1947. The present paper describes the devices for producing the voltage $KE^{\beta t}\cos\alpha t$ and of detecting α and β. *H. B. Curry.*

Eckart, Gottfried, et Kahan, Théo. Résolution d'une équation transcendante au moyen de la représentation conforme. Revue Sci. 86, 723–726 (1948).

This note is concerned with the positive roots of the equation

$$\frac{(\lambda^2-k_1^2)^{\frac12}-(\lambda^2-k_2^2)^{\frac12}}{(\lambda^2-k_1^2)^{\frac12}+(\lambda^2-k_2^2)^{\frac12}}=-\exp\{2h(\lambda^2-k_1^2)^{\frac12}\}$$

($0<k_1^2<k_2^2$, $h>0$), which appears in work on wave guides. A graphical method for the determination of these roots, based on properties of two elementary conformal maps, is indicated. *Z. Nehari* (St. Louis, Mo.).

Grohne, Diether. Rechenverfahren zur Auflösung von Gleichungssystemen. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig 1946, no. 4, i+28 pp. (1946).

Die n Gleichungen $u_i(x_1, \cdots, x_n)\equiv u_i=0$, $i=1, \cdots, n$, mögen die Lösungen x_i^* haben und man kenne einen Teilraum, in welchem x^* liegt. In ihm bildet Verfasser die Folgen

$$x_i^{(m+1)}=x_i^{(m)}+\sum_{k=1}^{m}A_{ik}u_k^{(m)}, \qquad i=1, \cdots, n.$$

Die Folgen konvergieren sicherlich, wenn $A_{ik}=-\partial x_i^*/\partial u_k$ und hinterher alle $u_i=0$ gesetzt werden. Da dies auf die Bestimmung der Tangentialebene der Fläche $u_i(x_1, \cdots, x_n)$ im Punkte x^* hinauskommt, ersetzt er diese Tangentialebene durch eine Sekantenebene in einem zu x^* benachbarten Punkt. Ein weiterer Abschnitt gibt einen Vergleich dieser Methode mit der regula falsi. Letztere konvergiert rascher, erfordert aber mehr Arbeit. Der 4-te Abschnitt bespricht ein graphisches Verfahren für zwei Unbekannte, der letzte Abschnitt endlich die höhere inverse Interpolation für eine Unbekannte. *E. Bodewig* (Den Haag).

Bílý, Josef. Solution of a system of linear equations with large coefficients in the diagonal. Aktuárské Vědy 8, no. 3, 114–127 (1949).

Verfasser gibt neue Beweise der Ergebnisse von Mises und Pollaczek-Geiringer [Z. Angew. Math. Mech. 9, 58–77 (1929)]. Neu ist die Darstellung eines Zyklus von n Seidel'schen Iterationen durch ein Produkt von n Matrizen. Damit beweist er den [schon früher von H. Geiringer gefundenen] Satz, dass auch die Seidel'sche Iteration konvergiert, wenn die Summe der absoluten Werte in jeder Spalte (Zeile) der Matrix kleiner als 1 ist [vgl. Geiringer, Reisner Anniversary Volume, pp. 365–393, Edwards, Ann Arbor, Mich., 1948; diese Rev. 10, 574]. *E. Bodewig.*

Banachiewicz, Tadeusz. Fragmentos de novo algorithmo de methodo de minimo quadratos. Rocznik Astr. Obserw. Krakow. Suppl. Internat. No. 20, 87–98 (1949). (Polish and Latino)

[The second language is the Latino sine flexione of Peano.] This paper exalts the alleged advantages of Cracovians for the solution of normal equations for the method of least squares. Two ways of using Cracovians are proposed, (A) the square root method based on the Cracovian equation $R^2=A$, A being in effect the augmented square matrix of the normal equations, and (B) the method of proportional factors based on the equation $G\cdot H=A$, where each kth line of H follows from the kth line of G by division of its elements by G_{kk}. *W. E. Milne* (Corvallis, Ore.).

Todd, John. The condition of a certain matrix. Proc. Cambridge Philos. Soc. 46, 116–118 (1950).

For the matrix A having all $a_{ii}=-2$ and all $a_{i,i+1}=a_{i+1,i}=1$ the author finds that the three measures for the ill-conditioning of A: Turing's N-number and M-number and the Wittmeyer-Neumann P-number (given by the ratio of the greatest and least eigenvalue of A) agree substantially. *E. Bodewig* (The Hague).

Bottema, O. A geometrical interpretation of the relaxation method. Quart. Appl. Math. 7, 422–423 (1950).

Das Gleichungssystem mit der Matrix A habe die Näherungslösung y. Dann findet man eine verbesserte Lösung durch Projektion des Vektors y auf den R_{n-1} mit der Gleichung $a_{i1}x_1+\cdots+a_{in}x_n=0$. *E. Bodewig* (Den Haag).

Fruchter, Benjamin. Note on the computation of the inverse of a triangular matrix. Psychometrika 14, 89–93 (1949).

Contains a known method for inverting a triangular matrix. *E. Bodewig* (The Hague).

✱Faure, Gérard, Simon-Suisse, J., et Rona, Th. Deux circuits analogiques pour l'inversion des matrices symétriques et la recherche de la vitesse critique de flutter. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 81–95.

Greville, T. N. E. Remark on W. M. Kincaid's "Note on the error in interpolation of a function of two independent variables." Ann. Math. Statistics 21, 137–138 (1950).

This is a correction to the review [these Rev. 9, 470] of the cited paper [same Ann. 19, 85–88 (1948)]; cf. these Rev. 10, 855.

Sponder, Erich. Construction graphique de la tangente en un point d'une courbe. Elemente der Math. 4, 86–88 (1949).

Um in einem Punkte P einer Kurve eine Tangente graphisch zu ziehen, schlage man um P mit mässig grossem Radius a einen Kreis, der die Kurve in den Punkten L (links von P) und R (rechts von P) schneidet. Mit einem Papierstreifen messe man die Differenz h der Bögen PL und PR: $\pm h = PL - PR$. Dann trage man in L die Strecke $8h/3$ senkrecht zur Sehne LR nach oben bzw. unten ab, verbinde den neuen Punkt O mit R und ziehe durch P die Parallele zu OR. Sie ist die Tangente in P. *E. Bodewig.*

Völlm, Ernst. Über die Rektifikation eines Kurvenbogens. Elemente der Math. 5, 4–7 (1950).

Ist der Kurvenbogen $\widehat{AC}$ monoton gekrümmt, ist ferner B der Schnittpunkt der Tangenten in A und C, so ist $\widehat{AC} = (2s+t)/3$, wo $s = \overline{AC}$, $t = \overline{AB} + \overline{BC}$, ein Näherungswert vierter Ordnung. *E. Bodewig* (Den Haag).

Reiz, Anders. On quadrature formulae. Proc. Cambridge Philos. Soc. 46, 119–126 (1950).

The author points out the great advantage in accuracy that Gauss's formulas possess in comparison to the Newton-Cotes formulas, and at the same time the great disadvantage due to the laborious interpolation required for irrational values of the argument. He then presents in tabular form the coefficients and values of the argument for a set of formulas that as far as possible retain the advantages and evade the disadvantages of Gauss's formulas. Altogether six sets of formulas are given, a set of eight with weight function equal to unity, a set of seven with weight function $(1-x^2)^{-\frac{1}{2}}$, seven with weight $(1-x)^{\frac{1}{2}}$, four with weight $(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$, four with weight e^{-x} and seven with weight e^{-x^2}. *W. E. Milne* (Corvallis, Ore.).

Bouzitat, Jean. Sur l'intégration numérique approchée par la méthode de Gauss généralisée et sur une extension de cette méthode. C. R. Acad. Sci. Paris 229, 1201–1203 (1949).

The author considers an extension of Gauss's generalized method of numerical integration to the case where two of the abscissas are chosen beforehand (say as the end points of the interval of integration) and the remaining points are determined so as to make the formulas exact for polynomials of as high degree as possible. He obtains the formulas in the case of the weight function $x^\alpha(1-x)^\beta$ for the interval $(0, 1)$, the weight $x^\alpha e^{-x}$ for the interval $(0, \infty)$, and weight $e^{-x^2/2}$ for $(-\infty, \infty)$. *W. E. Milne* (Corvallis, Ore.).

Tollmien, W. Über das Restglied der Mittelwertformeln für angenäherte Quadratur. Z. Angew. Math. Mech. 29, 193–198 (1949). (German. Russian summary)

Die Mittelwertsverfahren zur angenäherten Quadratur approximieren das Integral $\int_0^1 F(u)du$ durch ein lineares Aggregat von Werten des Integraden für einige ausgezeichnete Argumente u_ν $(\nu = 1, \cdots, n)$:

$$\int_0^1 F(u)du \approx \sum_{\nu=1}^n A_\nu F(u_\nu).$$

Die Gewichte A_ν werden nun so bestimmt, dass dieser Formel Polynome bis zum grade $m-1$ exact integriert, also

$$\sum_{\nu=1}^n A_\nu u_\nu^p = 1/(p+1), \qquad p = 0, 1, \cdots, m-1.$$

Bei den Quadraturformeln vom Cotesschen Typ ist $m = n$ oder $n+1$, bei den Gaussschen Formeln ist $m = 2n$. Der Verf. bestimmt

$$\int_0^1 F(u)du - \sum_{\nu=1}^n A_\nu F(u_\nu) = R_k$$

in exacter Integralform, wo R_k Ableitungen hochstens k-ter Ordnung enthalten soll. Er findet

$$R_k = \frac{1}{k!}\int_0^1 F^{(k)}(z)\sum_{\nu=1}^n A_\nu \bar{B}_k(u_\nu - z)dz, \qquad k = 1, 2, \cdots, m-1.$$

Die Funktionen $\bar{B}_k(v)$ sind die periodischen Bernoullischen Funktionen: $\bar{B}_k(v) = B_k(v)$, $0 \leq v < 1$; $\bar{B}_k(v+1) = \bar{B}_k(v)$ für alle v. Anwendung auf die Simpsonsche Formel liefert

$$R_4 = -\frac{1}{2880}[F^{\text{iv}}(1) - F^{\text{iv}}(0)] + \frac{1}{72}\int_0^1 F^{\text{iv}}(z)\{B_4(z) + 2\bar{B}_4(\tfrac{1}{2} - z)\}dz.$$

Bei der Trapezregel erhalt man die Euler-Maclaurinsche Quadraturformel mit dem Restglied

$$\frac{1}{(2\lambda+2)!}\int_0^1 F^{(2\lambda+2)}(z)\{B_{2\lambda+2}(z) - B_{2\lambda+2}\}dz.$$

S. C. van Veen (Delft).

Rubbert, F. K. Zur Praxis der numerischen Quadratur. Z. Angew. Math. Mech. 29, 186–188 (1949).

The determination of the errors in the formulas for numerical quadrature is too complicated if made by the usual formulas for the remainder. The author therefore estimates the error by choosing different kinds of formulas which include the integral and for which the relation of the errors is known. So he has the following combinations of formulas: (1) Simpson and Kowalewski; (2) Turazza and Gauss; (3) Simpson and Cotes; (4) Cotes and Sheppard. *E. Bodewig* (The Hague).

Timman, R. The numerical evaluation of the Poisson integral. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F. 32, i+16+ix pp. (1948).

The numerical evaluation of the Poisson integral

$$\tau(\varphi) = -(2\pi)^{-1}\int_0^{2\pi} \sigma(\psi)\cot\tfrac{1}{2}(\varphi-\psi)d\psi$$

is accomplished by dividing the interval of integration into 36 subintervals, in each of which the function $\sigma(\psi)$ is represented by an interpolating polynomial, just as in the case of ordinary numerical integration. Then the individual subintegrals, with $\sigma(\psi)$ replaced by a polynomial, have to be computed. This is done by means of series expansions. The problem requires that the successive interpolating polynomials join together continuously and with continuous derivatives. This is accomplished by the use of a method of interpolation due to Schoenberg. Finally the complete process is applied to illustrative examples and the results compared with those of other methods. *W. E. Milne.*

Athen, H. Genauigkeitssteigerung beim Beilschneidenplanimeter. Z. Angew. Math. Mech. 29, 375–377 (1949).

It is claimed that the simple hatchet planimeter [see Meyer zur Capellen, Mathematische Instrumente, 2d ed., Akademische Verlagsgesellschaft, Leipzig, 1944, p. 268; these Rev. 9, 160] gives measurements that have less variance than the more elaborate and more expensive planim-

eters. Its use has been hindered by the inherent systematic errors, but these can be reduced by the application of correction techniques. If the area of a circle is measured twice by traversing the circumference in opposite directions, one of the components of the error is eliminated by taking the mean measurement. The remaining error is a function of the ratio $r=R/L$, where R is the radius of the circle and L is the length of the arm of the hatchet planimeter. A table is given for corresponding values of r and the measured areas; this shows that the error is less than 0.1% for $r=0.10$ but is more than 0.7% for $r=0.5$. This table is used to reduce the errors of other measured areas by measuring with two different arms and applying weighting factors to obtain a more reliable mean. *M. Goldberg.*

Berthod-Zaborowski, Mme. Henri, et Mineur, Henri. Sur le calcul numérique des intégrales doubles. C. R. Acad. Sci. Paris **229**, 919–921 (1949).

This note deals with the numerical evaluation of a double integral by the method of iterated integration and the use of Gregory's or Gauss's formulas for the evaluation of the single integrals involved. Difficulties may arise due to the form of the region over which the double integration is to be performed. The authors treat three cases, showing how the difficulties can be circumvented. *W. E. Milne.*

Radon, Johann. Zur mechanischen Kubatur. Monatsh. Math. **52**, 286–300 (1948).

This paper deals with the extension of Gauss's method of mechanical quadrature to double integrals. A method is developed to determine, for an arbitrary region of integration, whether a formula of the desired type exists, and if it does, to set up the formula. The author gives a series of cubature formulas which are in part new and are presumably well suited for applications. A way of estimating the error is likewise shown. *W. E. Milne* (Corvallis, Ore.).

Charp, S. Fourier coefficient harmonic analyzer. Elec. Engrg. **68**, 1057 (1949).

A harmonic analyzer which computes Fourier coefficients by evaluation of their defining integrals, using mechanical ball-and-disc integrators, and presenting the results directly on counters. No change gears are required for computations up to the 100th harmonic. *S. H. Caldwell.*

✶Ostrowski, A. La recherche des périodicités cachées. Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 93–95. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

This paper gives brief mention of various methods for finding hidden periodicities. *W. E. Milne.*

✶Levy, H., and Baggott, E. A. Numerical Solutions of Differential Equations. Dover Publications, Inc., New York, N. Y., 1950. viii+238 pp. $3.00.

This was originally published as Numerical Studies in Differential Equations, Watts, London, 1934.

Stüssi, Fritz. Numerische Lösung von Randwertproblemen mit Hilfe der Seilpolygongleichung. Z. Angew. Math. Physik **1**, 53–70 (1950).

The author makes use of the string-polygon for a loaded beam to derive formulas for the numerical solution of certain differential equations of the second order. It is interesting to note that a number of the formulas so obtained are identical with some of the simpler mathematical formulas based on polynomial approximation. An example is

$$y_{m+1}-2y_m+y_{m-1}=\tfrac{1}{12}x^2(y''_{m+1}+10y''_m+y''_{m-1}).$$

W. E. Milne (Corvallis, Ore.).

✶Allen, D. N. de G. La méthode de libération des liaisons et les problèmes de charpentes. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 11–15. Centre National de la Recherche Scientifique, Paris, 1949.

✶Allen, D. N. de G. La méthode de libération des liaisons et la résolution des équations différentielles. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 16–17. Centre National de la Recherche Scientifique, Paris, 1949.

✶Allen, D. N. de G. Compléments pour l'application de la méthode de libération. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 18–34. Centre National de la Recherche Scientifique, Paris, 1949.

✶Vogel, Théodore. Le problème aux vibrations propres et la méthode de l'escalier. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 35–39. Centre National de la Recherche Scientifique, Paris, 1949.

Ludeke, Carl A. An electro-mechanical device for solving non-linear differential equations. J. Appl. Phys. **20**, 600–607 (1949).

A vibrating beam structure with nonlinear restoring force was constructed. This structure behaves in accordance with the relatively simple nonlinear differential equation

$$Id^2\theta/dt^2+\beta d\theta/dt+f(\theta)=P_0\sin\omega t.$$

Some discussion is given of the nature of the structural elements which might be used to represent the additional nonlinear terms in the equation

$$d^2\theta/dt^2+\psi_1(\theta,t)(d\theta/dt)^n+\psi_2(\theta,t)=\psi_3(t),$$

but there is no report of the actual construction of such elements. *S. H. Caldwell* (Cambridge, Mass.).

Meyerott, R. E., and Breit, G. A small differential analyzer with ball carriage integrators and Selsyn coupling. Rev. Sci. Instruments **20**, 874–876 (1949).

The machine described uses ball-and-disc integrators but depends upon the torque developed in selsyn units for the transmission of mechanical rotations. In order to avoid error in the transmission to integrator displacements a large step-down gear ratio is used. Such ratios lead to very long solution times unless the integrator discs are operated at high speeds. The authors do not indicate any provision for scale-changing gears. *S. H. Caldwell* (Cambridge, Mass.).

Walther, Alwin, und Dreyer, Hans-Joachim. Die Integrieranlage IPM-Ott für gewöhnliche Differentialgleichungen. Naturwissenschaften **36**, 199–206 (1949).

This is an account of a differential analyzer under construction in Germany since 1938 by the Institut für prak-

tische Mathematik (IPM) der Technischen Hochschule, Darmstadt, and the firm A. Ott. This is an analogue machine with (as yet) two integrators, a multiplier-divider, eight adders, and six function tables. The various components are connected electrically; and the machine is set up by plugging in wire connections on one of its units. The units are detachable so that the machine has a certain portability. The integrators work on the principle of a wheel rolling so as to describe the integral curve; the multiplier is a similar triangles device. Automatic curve followers, of a photo-electric type, and similar servomechanisms are used freely. The accuracy claimed under favorable circumstances is 0.1 per cent of the total ordinate-range.

H. B. Curry (State College, Pa.).

✶Barrois, W., et Simon-Suisse, J. Mecanisation des problèmes de vibrations et de flutter par le calcul matriciel. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 63–80.

✶Fairthorne, R. A. Digital machines for variational problems. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 331–334.

Bereis, Rudolf. Mechanismen zur Verwirklichung der Joukowsky-Abbildung. Arch. Math. 2, 126–134 (1950).

Joukowsky employed conformal transformations to obtain the stream lines and equipotential lines in aerodynamic flow. One of these is given by $z'=z+a^2/z$. The author mechanizes this transformation by a double Peaucellier cell of twelve links. The ends of the cell slide along the real and imaginary axes. By cutting the linkage in half, the number of links is halved to six without loss of operability. Several other variants of this mechanism are exhibited. In one of them, derivable from the Hart inversor, the parameter a is adjustable.

M. Goldberg (Washington, D. C.).

Pentkovskiĭ, M. V. A nomographic method of finding the best projective transformation of rectilinear scales. Doklady Akad. Nauk SSSR (N.S.) 66, 339–342 (1949). (Russian)

Let $s=s(x)$ be the equation of a scale, where s is arc length on the carrier, and let Δx, the increment between a plotted value of x and the next plotted value, be a given function of x. Then the quantity $\Delta\xi=s'(x)\Delta x/\{s(x_2)-s(x_1)\}$, where x_1 and x_2 are the limits within which the scale is to be constructed, is called the characteristic of the scale. In case Δx or $x^{-1}\Delta x$ is constant the characteristic is said to be uniform or logarithmic. If a scale for $f(x)$ which is carried on a straight line has been normalized by introducing $F(x)=[f(x)-f(x_1)]/[f(x_2)-f(x_1)]$ and subjected to a projective transformation, its characteristic is

$$\Delta\xi=(\mu+1)|F'(x)|\Delta x/[F(x)\mu+1]^2,$$

where μ is the parameter of the transformation when x_1 and x_2 are invariant. The author considers as best that value of μ for which the least value of $\Delta\xi$ is greatest. To find this value for straight scales an alignment chart with a curve net for μ and $\Delta\xi$ is presented and explained. Before using it, two parallel scales must be calibrated with $F(x)$ and $\Phi(x)=\{|F'(x)|\Delta x\}^{\frac{1}{2}}$, respectively. Some properties of the characteristic are stated and an analytic solution is given for μ in the case of the uniform and the logarithmic characteristic.

R. Church (Annapolis, Md.).

Nikolaev, P. V. On the projectivity of nomograms of M-functions. Doklady Akad. Nauk SSSR (N.S.) 67, 421–423 (1949). (Russian)

The set of functions $\varphi_1(t_1), \cdots, \varphi_r(t_1)$ is called a representing system in t_1 of dimension r for the real function $F(t_1, \cdots, t_n)\equiv F(t)$ if, for all α_j in an n-dimensional domain G for which the function $F(t)$ is defined, $F(t_1,\alpha_2,\cdots,\alpha_n)\equiv F(t_1,\alpha)$ is a linear combination of the $\varphi_j(t_1)$. Then $F(t)$ is said to be nomographically rational in t_1. The $\varphi_j(t_1)$ are called a basis in t_1 for $F(t)$ if it is impossible to construct out of linear combinations of the $\varphi_j(t_1)$ a representing system of lower dimension. Then $F(t)$ is said to be of dimension r in t_1. If the $\varphi_j(t_1)$ are linearly independent there exists a unique representation: $F(t)\equiv\sum_{j=1}^r F_j(t_2, \cdots, t_n)\varphi_j(t_1)$, and if the coefficients $F_j(t_2, \cdots, t_n)$ are also linearly independent the above representation is said to be basic in t_1. If $F(t)$ is equal to an nth order Massau determinant $|f_{i1}(t_i); \cdots; f_{in}(t_i)|$ and is not degenerate (i.e., not the product of factors each of which contains fewer than all n variables) it is said to be an M-function.

The paper extends to n variables some of the author's work for three variables [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 582–584, 774–777 (1940); these Rev. 2, 240] and proves that if for an M-function of dimension n in at least one variable there exists a nomogram in which at least one carrier C_i does not belong to any hyperplane of the $(n-1)$-dimensional space in which the C_i lie, then all nomograms of this function are projective. Examples for three variables in earlier work show that the result is not true if the dimension is less than n for all variables.

R. Church.

Nikolaev, P. V. Anamorphosis of a function. Doklady Akad. Nauk SSSR (N.S.) 68, 229–232 (1949). (Russian)

The investigations reviewed above are continued with a view to obtaining conditions on $F(t)$ in order that it may be an M-function. If $F(t)$ has a basis in t_1 and in t_2, there exists a decomposition

$$F(t)\equiv\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\varphi_{1i}(t_1)\varphi_{2j}(t_2)v_{ij}(t_3, \cdots, t_n),$$

where r_k is the dimension of $F(t)$ in t_k. The matrix $T^{(12)}\equiv\|v_{ij}\|$, called an anamorphosis matrix (A-matrix) of $F(t)$, is said to be basic if the above decomposition is basic in t_1 and t_2. Assuming that the dimension of $F(t)$ in each t_k is not greater than n and is equal to n for t_1 and that $F(t)$ is not degenerate, the principal result is that $F(t)$ is equal to a Massau determinant if and only if for each of the $(n-1)$ basic A-matrices $T^{(1k)}$ $(k=2, \cdots, n)$ there exists a matrix of constants C_k such that $T^{(1k)}C_k$ is skew-symmetric (when $r_k=n$) or truncated skew-symmetric (when $r_k<n$). The representing system $\varphi_{1i}(t_1)$ in t_1 (for example) may be obtained by choosing from the family of functions $F(t_1, \alpha)$ a set which are independent and maximum in number. An example for $n=3$ is given showing how the A-matrix and M-determinant may be determined in practice. It is stated that the results may be generalized to give conditions for representation of $F(t_1, \tau_1, \cdots, t_n, \tau_n)$ as a binary Massau determinant.

R. Church (Annapolis, Md.).

Boulanger, Georges. Sur la notion de contact nomographique. C. R. Acad. Sci. Paris 229, 971–973 (1949).

The nomograms considered consist of a number of superposed transparent planes on each of which are drawn num-

bered curves or curves of numbered points. A contact consists in the tangency of two curves or the passage of three curves through a point. The author states the existence of 100 types of contact, among which he describes 58 as usable. Each usable contact can be decomposed into contacts falling into 18 classes called fundamental. *J. M. Thomas.*

ASTRONOMY

***Stumpff, K. Himmelsmechanik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 20, pp. 43–74. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This is a collection of reviews of publications that appeared in Germany or by German authors during the indicated period. Over one-half of the collection deals with articles on the two-body problem and ephemeris calculation. In addition there are reviews of investigations on the development of the disturbing function, on the resonance problem in planetary motions and on graphical and numerical solutions. *D. Brouwer* (New Haven, Conn.).

Baženov, G. M. On some applications of matrices in celestial mechanics. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. **4**, no. 4(57), 143–168 (1949). (Russian)

The paper deals with the application of the so-called "Cracovian" matrices first used by Banachiewicz, to a varied assortment of problems of celestial mechanics. The problems are: transformation of celestial coordinates, precession, restricted three-body problem, perturbations of orbital elements, canonical elements, improvement of orbits and computation of perturbed elements. Some of these items have been treated by various authors before and in a similar manner, in connection with specific astronomical problems, but most of the existing literature on the subject is fragmentary. *L. Jacchia* (Cambridge, Mass.).

Porter, J. G. The differential correction of orbits. Monthly Not. Roy. Astr. Soc. **109**, 409–420 (1949).

In effect an adaptation of Tietjen's method to calculating machines. Particular attention is given to reducing the labor of solving the equations for the differential corrections to the orbital constants. *D. Brouwer.*

Merton, G. A modification of the perturbations-of-elements method. Monthly Not. Roy. Astr. Soc. **109**, 421–435 (1949).

The investigation deals with the method of numerical integration applied to the perturbations in the elements. The modification introduced amounts, in effect, to the use of the perturbed mean anomaly as independent variable. It is shown that this permits taking advantage of special tables which have been constructed for the usual form of the method, while any desired order of accuracy may be attained. *D. Brouwer* (New Haven, Conn.).

Pavel, F. Bestimmung einer Doppelsternbahn mit 90° Neigung. Astr. Nachr. **277**, 153–157 (1949).

The author develops formulae necessary for the evaluation of elements of visual binary systems whose orbital planes are perpendicular to the celestial sphere. By differentiating the fundamental equations, the author establishes, furthermore, a set of linear equations furnishing the differential corrections to the previously adopted values of the elements, which may be required to establish the best possible fit to the observations. These equations are applied to the binary system of 42 Comae worked out as an illustrative example, and new elements of this visual binary are derived which differ but little from those computed previously by Haffner [Astr. Nachr. **276**, 145–156 (1948)]. *Z. Kopal* (Cambridge, Mass.).

Agostinelli, Cataldo. Sullo spostamento dei perieli dei pianeti. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 21–31 (1949).

Report of a lecture.

Agostinelli, Cataldo. Sulla variazione degli elementi ellittici dell'orbita di un pianeta attratto dal sole con una legge analoga a quella di Weber. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 167–189 (1949).

The author treats the motion of a planet about the sun assuming a potential function for two point masses of the form $U = kmm'r^{-1}(1+\dot{r}^2/c^2)$, where c is the velocity of light, and considering the sun to be an oblate rotating body. The work is carried out with a free use of approximations to arrive at explicit formulae for the variation of the elliptic elements of the orbit. In particular, if a foreshortening of the sun's polar diameter of .05% is assumed the observed value for the advance of Mercury's perihelion is obtained. *R. G. Langebartel* (Urbana, Ill.).

Milankovitch, M. Ueber die Verwendung vektorieller Bahnelemente in der Störungsrechnung. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 1–70 (1939).

The principal object of this investigation is to develop with the aid of vector methods equations for the variation of elliptic elements. A vector perpendicular to the orbital plane (the integral of areas) and a vector with modulus μe directed toward the perihelion serve to define five independent elements. As the sixth element the time of perihelion is used. By this particular choice of elements a considerable degree of symmetry is gained. The second part of the investigation deals with the derivation of the Lagrangian brackets and the resulting equations of the variations for these elements. The author finally derives from his results the equations of the variations of the usual elliptic elements. *D. Brouwer* (New Haven, Conn.).

***Milankovitch, M. Kanon der Erdbestrahlung und seine Anwendung auf das Eiszeitenproblem.** Académie Royale Serbe. Éditions Speciales, Tome CXXXIII. Section des Sciences Mathématiques et Naturelles, Tome 33. Belgrade, 1941. xx+633 pp.

Contents: Preface. Part I: The movements of planets around the sun and their mutual perturbations. Chapter 1: Newton's law of gravitation. 2: Two body problem and the unperturbed heliocentric movement of the planets. 3: The perturbations. Part II: Different kinds of rotation of the earth. 4: Some necessary theorems and equations. 5: The diurnal rotation of the earth and its consequences. 6: Precession. 7: Astronomical nutation of the earth's axis. 8: Time and chronology. Part III: Secular movements of the earth's poles. 9: Mathematical formulation of the problem. 10: The dynamical asymmetry of the earth's crust and its consequences. 11: The mechanism of the secular movements of the poles. 12: Numerical and cartographical rep-

resentation of the secular movements of the poles. Part IV: The insolation of the earth and its secular variations. 13: The present state of insolation. 14: Secular variations of the astronomical elements affecting the insolation. 15: Analytical representation of the secular variation of the insolation. Part V: Connection between the insolation, temperature of the earth and its atmosphere. The mathematical climate of the earth. 16: Introduction. The basic facts and laws. 17: Connection between a steady insolation and the earth's temperature. 18: Connection between a variable insolation and the earth's temperature. Part VI: The glaciations, their mechanism, classification and chronology. 19: The history of the problem of ice ages. 20: The course of the insolation during the last 600000 years. 21: The insolation curves as basis for the chronology and the assignment of the ice ages. 22: The climatic effects of the secular course of the insolation. 23: The mechanism and course of the large climatic oscillations in the Quaternary.

This work represents a synthesis of the author's investigations, published before in many papers. The aim of the first two parts is to give in a vectorial form the astronomical and mechanical material necessary for the understanding of the theories developed in the other parts. The introduction of a new system of astronomical elements and a project for calendar reform are the author's two principal contributions to the questions treated in these parts. The author's theory of the displacements of the earth's poles during the geological periods is given in part III. The possibility of the poles' movements is derived from the asymmetrical distribution of the masses in the earth's crust and the theoretical results are compared with the paleoclimatic data.

Parts IV–VI contain a full presentation of the author's theory of the glaciations. The mathematical theory of the earth's climate has shown that the variations of the astronomical elements: the eccentricity of the earth's path, the obliquity and the length of the perihelion, produce an important effect on the changes of the climate. The well-known curves of Milankovitch representing the changes in climate give an explanation of the ice ages during the last 600000 years. The author also discusses the objections which have been raised against the theory and the investigations confirming it. *W. Jardetzky* (New York, N. Y.).

Hölder, Ernst. Zusätzliche Stabilitätsbetrachtung betreffend "Die symmetrischen periodischen Bahnen des restringierten Dreikörperproblems in der Nachbarschaft eines kritischen Keplerkreises." Amer. J. Math. **72**, 157–160 (1950).

The author considers the stability of the periodic orbits determined in his earlier paper [Amer. J. Math. **60**, 801–814 (1938)]. Let κ be the difference between the Jacobian constants for the periodic orbit and a (critical) Kepler circle, and let ξ be the $(m+1)$th Fourier coefficient (m the Hill periodic ratio) in the development of the radius vector of the periodic orbit in a cosine series on multiples of the elongation. Then the region in the (κ, ξ) plane between the two curves (to leading terms only) $\kappa=-3\nu\xi^2$, $\kappa=-\nu\xi^2$, where ν is a positive parameter depending on m and the Kepler circle radius, is a region of instability.

R. G. Langebartel (Urbana, Ill.).

✶Heckmann, O. Statistische Dynamik von Sternsystemen. Naturforschung und Medizin in Deutschland 1939–1946, Band 20, pp. 393–411. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Gialanella, Lucio. Sul problema dei due corpi di masse variabili in cui la forza attrattiva è proporzionale alla distanza. Applicazione agli ammassi globulari di stelle. Mem. Soc. Astr. Ital. (N.S.) **20**, 93–105 (1949).

The problem of the disintegration of massive globular clusters has wide cosmogonical significance. The author is concerned with the model of a homogeneous globular cluster in which the density is a decreasing function of the time. For the case in which the total mass of the cluster decreases as $1/t^2$, the tendency of the cluster toward disintegration is proportional to the fourth root of the rate of loss of mass. The reviewer doubts the validity of the model. The law of force is not even approximately one of proportionality to the distance from the center and the escape of stars would more likely lead to an increase in average density, coupled with a reduction of the cluster's radius, than to a decrease in the density. *B. J. Bok* (Bloemfontein).

Bucerius, H. Integralgleichungstheorie des Sternaufbaus. IV. Über rotierende Polytropen. Astr. Nachr. **275**, 49–72 (1947).

[For part III cf. Astr. Nachr. **267**, 252–271 (1939).] The equilibrium of a gravitational mass with polytropic pressure-density relation is considered. A stationary potential field from external sources is supposed to be present. The method consists in finding equipotential surfaces of the total combined potentials. Such surfaces then represent boundaries of possible equilibrium configurations. In a first section the classical solutions for incompressible fluids are shown to be included in this general study. Also the rotating Laplace polytropes are treated. In later sections the problem is treated by the methods of integral equations. The general results of this study are finally applied to study the rotational symmetrical configurations branching off from the Maclaurin ellipsoid. *G. Randers* (Oslo).

Burman, U. R. On the possibility of fitting of convective cores of different compositions to a given envelope solution of the stellar equations. Proc. Nat. Inst. Sci. India **15**, 269–275 (1949).

It is known that stellar models are possible in which a convective core is fitted to the radiative envelope. The question which the author considers is whether such a core is uniquely determined by the corresponding envelope. The problem is restricted to the case of uniform molecular compositions in each of the two separate regions of the star. A discontinuous transition of molecular weight on the border of the core is, however, allowed. It is shown that with this restriction it is not possible to fit more than one core to the given envelope solution. This would then support the belief that the calculated models of known stars give a unique solution for the central core. This conclusion rests also on the assumption made in the paper that the hydrogen content is smaller in the central core than in the envelope.

G. Randers (Oslo).

Ghosh, N. L. Gases with a pressure-density relation. Impossibility of confocal density distributions. Bull. Calcutta Math. Soc. **41**, 183–186 (1949).

In earlier papers the author has proved certain restrictions on possible density distributions of equilibrium configurations of a rotating mass of gas. Here proof is given that, with a density depending on pressure alone, it is impossible to have a density distribution in confocal ellipsoids. This is proved both for constant and variable angular velocity.

G. Randers (Oslo).

Underhill, Anne B. Transfer problems in an atmosphere with continuous scattering and continuous absorption. Astrophys. J. 110, 340–354 (1949).

Wrubel, Marshal H. The transfer of radiation in a spherical atmosphere of electrons. Astrophys. J. 110, 288–303 (1949).

This paper considers the pair of integro-differential equations [cf. Chandrasekhar, same J. 103, 351–370 (1946); these Rev. 7, 494] which govern the transfer of radiation in the two states of polarization in a spherical atmosphere of free electrons scattering according to Thomson's laws. The equations are treated in an approximation which is equivalent to retaining terms up to $P_2(\mu)$ in an expansion in spherical harmonics and the case when the radial optical thickness (τ), measured from $r=\infty$ inward, varies as some inverse power of r is examined in detail. Solutions are found for the case when the atmosphere is bounded in the inside by a surface radiating in the outward directions in a known manner. The solutions are illustrated numerically and graphically for the case $\tau \propto 1/r$. *S. Chandrasekhar.*

Tuominen, Jaakko. On the appearance of vortex movements in the sun. Nederl. Akad. Wetensch., Proc. 52, 747–759 (1949).

The author sets out to investigate the possibility of a connection between the motion of sunspots on the sun's surface with possible vortex motion following from the differential rotation of the sun. He proves that, under certain assumptions and simplifying restrictions, the solution of the hydrodynamical equations of motion does not represent this kind of motion. The assumptions and restrictions are so extreme that the reviewer doubts whether any valuable conclusions can be drawn. The assumptions are, among others: rotation in cylindrical shells for the main rotation of the sun, streamlines stationary in fixed space, polytropic density distribution, temperature unaffected by the vortex motion, pure harmonic oscillation and predetermined frequency. *G. Randers* (Oslo).

Agekyan, T. A. On the dynamics of stellar growth through a cloud of meteoric material. Doklady Akad. Nauk SSSR (N.S.) 69, 515–518 (1949). (Russian)

The problem of accretion of the stellar masses through attraction of cosmic dust is examined quantitatively. No special assumption is made regarding the trajectory of the star with respect to the cloud and an arbitrary velocity distribution is assumed for the individual dust particles. The analysis provides the means of excluding all particles that collide with relative velocities larger than a critical value, for which they are turned into gas and thus repelled by the star's radiation pressure. The author has tried to solve the problem in the most general case and no attempt has been made to give any numerical results. Only for the sun are optimum conditions computed; these could presumably have led to the formation of a planetary system. *L. Jacchia* (Cambridge, Mass.).

RELATIVITY

Landsberg, P. T. Lorentz-like transformations. Nature 161, 208 (1948).

A class of transformations is considered which includes as subgroups the Lorentz transformations and the transformations of H. Dingle's theory of temperature radiation. *A. Schild* (Pittsburgh, Pa.).

Milne, E. A., and Whitrow, G. J. On the so-called "clock-paradox" of special relativity. Philos. Mag. (7) 40, 1244–1249 (1949).

In the clock paradox of special relativity, two observers who coincide, part company, and then coincide again, record different lapses of time while apart; the paradox being that in this argument the two observers are interchangeable. It is shown in the present paper that there is no paradox in the case of two observers, who are equivalent and have equivalent clocks in the sense of kinematical relativity, regardless of their relative motions. A more subtle form of the paradox is enunciated in a world model with spatial part of constant positive curvature, in which case there is no question of relative acceleration. It is concluded that the restriction of equivalence to the system of fundamental observers in such a world model is essential if paradoxes are to be avoided. This is an argument in favour of restricting the system of observers as in kinematical relativity. *A. G. Walker* (Sheffield).

Papapetrou, A. Drehimpuls- und Schwerpunktsatz in der relativistischen Mechanik. Prakt. Akad. Athēnōn 14, 540–547 (1939). (German. Greek summary)

The author considers the quantity $F_{\mu\lambda\kappa}=2l_{[\lambda}T_{\mu]\kappa}$, where $T_{\mu\lambda}$ is the material energy-tensor satisfying the equations $\partial_\mu T^{\mu\lambda}=0$ (no forces) and $l^\lambda=x^\lambda-\xi^\lambda$, ξ^λ being a fixed point in the space-time world. The equations $\partial_4 J_{\lambda\kappa}=0$, $J_{\lambda\kappa}=-c^{-1}\int F_{\lambda\kappa 4}dx^1dx^2dx^3$ express the theorem of the straightness of the world line of the centre of gravity and the theorem of conservation of the angular momentum. It is proved, however, that the centre of gravity and the angular momentum are not Lorentz invariant. The possible world lines of the centre of gravity form a cylinder. *J. Haantjes* (Leiden).

Papapetrou, A. Non-symmetric stress-energy-momentum tensor and spin-density. Philos. Mag. (7) 40, 937–946 (1949).

In special relativity the assumption that the energy-momentum tensor T_{ij} is symmetric is connected with the assumption that we consider material systems in which there is only orbital angular momentum. If we consider systems of spinning particles then the requirement of a symmetric T_{ij} is no longer a necessity. The author gives a detailed discussion of the physical meaning of a non-symmetric energy-momentum tensor and by classical means relates the anti-symmetric part of T_{ij} to the spin density. These ideas are then applied to the unified field theory of Einstein and Straus and it is shown that this latter theory does not give an explanation of geomagnetism. *M. Wyman* (Edmonton, Alta.).

Dirac, P. A. M. Forms of relativistic dynamics. Rev. Modern Physics 21, 392–399 (1949).

A fundamental problem of quantum theory is to combine the restricted relativity principle with the Hamiltonian formulation of dynamics. The first is satisfied by the requirement that physical laws shall be invariant under an infinitesimal Lorentz transformation, which is of the form

$$(1)\qquad u_\mu{}^*=u_\mu+a_\mu+b_\mu{}^\nu u_\nu,\quad b_{\mu\nu}=-b_{\nu\mu},\qquad \mu,\nu=1,2,3,4.$$

To satisfy the second it is necessary that any two dynamical variables ξ, η should have a Poisson bracket $[\xi,\eta]$,

satisfying the usual relations. Under coordinate transformations each variable ξ must change according to the law $\xi^* = \xi + [\xi, F]$, where F is some infinitesimal dynamical variable associated with the dynamical system and with the transformation (1). This leads us to write

$$F = -P^{\mu}a_{\mu} + \tfrac{1}{2}M^{\mu\nu}b_{\mu\nu}, \quad M^{\mu\nu} = -M^{\nu\mu}.$$

Dirac calls P^{μ}, $M^{\mu\nu}$ the ten fundamental quantities of the dynamical system. Of these P^0 is the total energy, and P^r, M^{rs} ($r, s = 1, 2, 3$) are respectively the total linear momentum and the total angular momentum. They satisfy the Poisson bracket relations

$$[P_{\mu}, P_{\nu}] = 0, \quad [M_{\mu\nu}, P_{\rho}] = -g_{\mu\rho}P_{\nu} + g_{\nu\rho}P_{\mu},$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\rho\mu}.$$

The problem of finding a new dynamical system reduces to finding a new solution of these equations.

Dynamical theory may be formulated in three different forms: (i) the instant form, which is the usual form where the dynamical variables refer to physical conditions at some instant of time, that is, over the three-dimensional surface $u_0 = 0$; (ii) the point form, which is invariant under the group of rotations round the point $u_{\mu} = 0$ and in which the variables refer to physical conditions over the surface $u^{\mu}u_{\mu} = k^2$ (k constant); and (iii) the front form, where the variables refer to conditions over a plane wave front, say, $u_0 - u_3 = 0$. The fundamental quantities are worked out in the three cases for a system of particles with interaction and for the electromagnetic field. *A. J. McConnell* (Dublin).

✶Landau, L., and Lifšic, E. Teoriya Polya. [Theory of Fields]. 2d ed. OGIZ, Moscow-Leningrad, 1948. 364 pp.

This book is concerned with two main subjects: the special-relativistic theory of the electromagnetic field, and the basic content of general relativity as a field theory of gravitation. The first two short chapters present the main ideas of special relativity and relativistic mechanics; the next seven chapters deal with the electromagnetic field; and the tenth and eleventh chapters, rather less than one-third the length of the book, contain the discussion of general relativity. The treatment of the electromagnetic field has as a special feature the use of variation principles as the basic method. The significance of the energy momentum tensor is fully expounded, and the general method for securing its symmetry when the canonical tensor is unsymmetrical is indicated. The use of multipole expansions is explained both for the static case and for a radiating system. As one further interesting addition to the material usually given there may be mentioned the inclusion of a chapter on the basic theoretical ideas of geometrical optics and of diffraction. The chapters on general relativity contain a discussion of the physical requirements for a relativistic theory of gravitation, and a development of the essential mathematical methods. The reduction to the Newtonian theory in the weak-field limit is given, and the precession of the perihelion and the deflection of light passing near the sun are calculated. The ordinary gravitational red-shift appears not to be treated; the cosmic red-shift is mentioned in connection with a brief discussion of the simplest cosmological models. *W. H. Furry* (Copenhagen).

Weyl, Hermann. A remark on the coupling of gravitation and electron. Physical Rev. (2) **77**, 699–701 (1950).
Weyl, Hermann. A remark on the coupling of gravitation and electron. Actas Acad. Ci. Lima **11**, 1–17 (1948).

[A footnote to the first paper states that it is an abbreviated version of the second paper, which "was made incomprehensible by numerous misprints."] It is known that the Einstein gravitational and the Maxwell electromagnetic field equations may be obtained from a variational principle involving a Lagrangian, in which both the metric tensor and the symmetric affine connection Γ^{r}_{pq} are taken as basic field quantities, and submitted to independent arbitrary infinitesimal variations, as well as from a variational principle involving a Lagrangian, which is a function only of the metric tensor and its derivatives, and only this tensor is varied. The author shows that if to this Lagrangian there is added a term which will give the Dirac equations for an electron on varying the wave functions the equivalence between the former method (the so-called mixed theory) of deriving the field equations and the latter method (the so-called metric theory) is lost. However, he shows that the mixed theory with the Lagrangian ϵH, where

$$H = R + K[L + W(l, l_2, M_1, Q)]$$

is identical with the metric theory based on the Lagrangian ϵH^*, where

$$H^* = R^* + K[L^* + W^*(l, l_2, M, Q) - \tfrac{3}{8}Kl_2].$$

The star denotes the fact that the functions involved are to be considered as functions of the $g_{\mu\nu}$ and their derivatives. The quantities l, M and l_2 are invariants formed from the wave functions ψ, the first two bilinear in ψ and $\bar{\psi}$, the last biquadratic in ψ and $\bar{\psi}$; Q is the invariant formed from the electromagnetic field tensor. *A. H. Taub* (Urbana, Ill.).

Finzi, Bruno. Il campo elettromagnetico nello spazio-tempo. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 127–144 (1949).

Report of a lecture.

Hund, F. Zugänge zum Verständnis der allgemeinen Relativitätstheorie. Z. Physik **124**, 742–756 (1948).

In der Newtonschen Mechanik lässt sich die Beschleunigung in einem beliebigen starren Bezugssystem in der Gestalt $F + (v/c) \times G$ schreiben, wo v der Geschwindigkeitsvektor ist, F der polare Vektor und $G = -2c\sigma$ (σ ist die vom Ort unabhängige Rotationsvektor des Bezugssystem) der axiale Vektor des Gravitationsfeldes ist. Die Vektoren F und G genügen einigen Feldgleichungen welche einige Ähnlichkeit zeigen mit den elektromagnetischen Feldgleichungen. Diese Gleichungen werden nun erweitert in einer Weise welche mit der Idee zusammenhängt, dass das Feld G durch den Umlauf ferner Massen entsteht. Die Wirbel von G werden durch die Bewegung von Materie bestimmt. Die Gleichungen dieser Modelltheorie sind nicht Lorentzinvariant aber einige wesentliche Punkte der Einsteinsche Gravitationstheorie sind schon zu erkennen. Die Theorie wird aber nicht gebracht als Stellvertreter der Einsteinschen Theorie aber als Ruhepunkt zwischen Newtonsche und Einsteinsche Theorie. *J. Haantjes* (Leiden).